CS 49/149: 21st Century Algorithms (Fall 2018): Problem Set 0

The problems below are not for submission. Rather, it indicates the mathematical maturity that a student should be comfortable with. That is, if a student feels that they can *tackle* these problems and is confident that given enough time can either solve or make considerable progress, then they should have no problems in the math that they will face in CS 49/149.

Also, we will be using some facts established below throughout the course.

Problem 1. Prove the following inequalities

1. (☕) For any real x, prove $e^x \geq 1 + x$.
2. (☕) For any real x, prove $\frac{e^x + e^{-x}}{2} \leq e^{x^2}$.
3. (☕☕) For any real $x < 0.5$, $e^x \leq 1 + x + x^2$.

Problem 2. Prove the following

1. (☕☕) For any $n > 0$, $\sum_{i=1}^{n} 1/i = \Theta(\log n)$.
2. (☕) For any $n > 0$, $\sum_{i=1}^{n} 1/i^2 = \Theta(1)$.

Problem 3. Stirling's approximation states that for any natural number n,

$$\sqrt{2\pi} \cdot \left(n^{n+\frac{1}{2}} e^{-n}\right) \leq n! \leq e \cdot \left(n^{n+\frac{1}{2}} e^{-n}\right)$$

(☕☕) Use this to prove $\binom{n}{n/2}/2^n = \Theta(1/\sqrt{n})$ (assume n is even).
(☕☕☕) (**Extra Credit:**) In fact prove that for any constant c,

$$\binom{n}{\frac{n}{2} \pm c\sqrt{n}}/2^n = \Theta(1/\sqrt{n})$$

Problem 4. (Probability Basics)

1. (☕) Prove for any *non-negative* random variable Z and $t > 0$,

$$\Pr[Z > t] \quad < \quad \frac{\mathbf{Exp}[Z]}{t}$$

   This is called Markov's inequality, or sometimes an *averaging argument*.

2. (☕) Prove for any random variable Z and $t > 0$,

$$\Pr[|Z - \mathbf{Exp}[Z]| > t] \quad \leq \quad \frac{\mathbf{Var}[Z]}{t^2}$$

   This is called Chebyshev's inequality or the second-moment inequality.

3. Now we establish a much stronger inequality for sums of *independent variables*. This is *super important* and such bounds are called Chernoff-Hoeffding-Azuma bounds. Knowing the result is important, but is also important to know how things are proved. We do this in steps.

   For $1 \leq i \leq n$, let X_i be a random variable which is $+1$ with probability $1/2$ and -1 with probability $1/2$. These are mutually *independent* random variables (recall what this is). Let $Z = X_1 + \cdots + X_n$ be the sum of these random variable.

(a) (☕) What is $\mathbf{Exp}[Z]$?

(b) (☕) What is $\mathbf{Var}[Z]$? What upper bound does the Chebyshev inequality say about $\Pr[Z > t]$ for $t > 0$?

(c) (☕) Now note that $\Pr[Z > t]$ is at most $\Pr[e^{\lambda Z} > e^{\lambda t}]$ for any $\lambda \geq 0$. Also note that $e^{\lambda Z}$ is a positive random variable. Apply Markov's inequality on it.

(d) (☕☕) What is $\mathbf{Exp}[e^{\lambda Z}]$? Recall what Z is, and recall that the X_i's are independent. Recall the relation between $\mathbf{Exp}[f(X)g(Y)]$ and $\mathbf{Exp}[f(X)] \cdot \mathbf{Exp}[g(Y)]$ for deterministic function f and g.

(e) (☕☕) Now can you optimize λ so that the RHS of what you get from part 3 (and putting in part 4's calculations) is minimized? What is the final answer you get?