Homework 6

Due: July 28th, 2009

In this assignment you will be asked to solve LPs. You could use any LP solver you wish; maybe you were taught one in CO350. There is an LP solver available as a Java applet on the web:
`http://www2.isye.gatech.edu/~wcook/qsopt/downloads/codes/java/applet.htm`

(Total 30)

1. (10 marks)
   Consider the following instance of $(R||C_{max})$.

	Job 1	Job 2	Job 3	Job 4
m/c 1	2	2	2	4
m/c 2	3	1	2	2
m/c 3	1	3	4	4

   (4) Write the integer program to solve the above instance.
   (3) Find the solution to the corresponding LP relaxation. Write the values of the variables and the optimal makespan returned by the LP. Also mention which solver you used.
   (3) Find the optimal solution to the IP by hand (its a small instance, you should be able to do it). What is the integrality gap for this instance?

2. (4 marks)
   Consider the following instance of $(1|prec|\sum w_j C_j)$. There are three jobs $(1,2,3)$ with weights $(5,12,7)$ respectively, and processing times $(3,4,2)$ respectively. The precedence digraph has 3 arcs as follows: $(1 \to 3), (2 \to 3)$ and $(1 \to 2)$. Write an LP relaxation to solve the above problem.

3. (6 marks) Find an example of an instance on 4 machines for which the optimum makespan of the flow shop problem is *strictly* smaller than the optimum makespan of the flow shop problem *with* the permutation constraint. That is, find an example such that

$$OPT(F4||C_{max}) < OPT(F4|prmu|C_{max})$$

4. (10 marks)
   Consider the following instance of $(J||C_{max})$.

Jobs	Machine Order	Processing Times
1	$1, 2, 3, 4$	$p_{11} = 9,\ p_{21} = 8,\ p_{31} = 4,\ p_{41} = 4$
2	$1, 2, 4, 3$	$p_{12} = 5,\ p_{22} = 6,\ p_{42} = 3,\ p_{32} = 6$
3	$3, 1, 2, 4$	$p_{33} = 10,\ p_{13} = 4,\ p_{23} = 9,\ p_{43} = 2$

   (a) **(3)** Draw the disjunctive graph corresponding to the above instance. Clearly mark the fixed arcs and the disjunctive edges (different colors, or solid and dotted lines).
   (b) **(4)** Orient the disjunctive edges in *any* way you think is optimal. Once you fix the orientation, find the corresponding schedule and the value of the makespan.
   (c) **(3)** Write the integer programming formulation for the above instance.

5. **Bonus** Let U be a set of n items. Recall $f : 2^U \rightarrow \mathbb{R}$ is a submodular function if for all subsets $S, T \subseteq U$,

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$$

We now establish that the function described in the lecture notes is submodular.

(a) **(1)** Given two submodular functions f and g, define $(f+g)$ to be $(f+g)(S) = f(S)+g(S)$. Show that $(f+g)$ is submodular.

(b) **(1)**Given $\{p_1, p_2, \ldots, p_n\}$, each $p_i \geq 0$ for all $i \in U$, define

$$f(S) := \sum_{i \in S} p_i$$

Show f is submodular.

(c) **(2)** Show that for any subsets $S, T \subseteq U$, and any $\{p_1, \ldots, p_n\}$, $p_i \geq 0$ for all $i \in U$

$$(\sum_{i \in S} p_i)(\sum_{i \in T} p_i) \geq (\sum_{i \in S \cup T} p_i)(\sum_{i \in S \cap T} p_i)$$

(d) **(2)**Using part(c) above, show that the function

$$f(S) = -\sum_{j \in S} p_j^2$$

is submodular.