Packet Routing with Fixed Paths. Input is a directed graph and k source sink pairs s_i, t_i **and** paths p_i for each pair. We need to **route** packets from s_i to t_i along these paths p_i. Suppose it takes one unit of time to traverse an edge (u, v) and at a time only one edge can use (u, v). Packets can wait at a vertex, however. We want to find a schedule which minimizes *makespan*: the time at which the last packet reaches its destination.

Let

$$C := \max_{e \in E} |\{i : e \in p_i\}| \quad \text{and} \quad D := \max_{i=1}^{k} |p_i|$$

be the *congestion* and *dilation* of the input. Observe that $OPT \geq \max C, D$, and so, $OPT = \Omega(C + D)$. We now show that there is a schedule with makespan $OPT \leq O(C + D\log(kD))$.

We first describe a generic "reduction" from an infeasible schedule to a feasible schedule. Call a schedule α-feasible for some positive integer α, if it sends at most α packets through any edge at any time. So $\alpha = 1$ would imply a feasible schedule.

Lemma 1. *Given an α-feasible schedule with makespan T, we can construct a feasible schedule with makespan $\leq \alpha T$.*

Proof. An easier proof of αT: Dilation of time. Instead of doing things at $t = 1, 2, \ldots$ do it in $t = 1, \alpha + 1, 2\alpha + 1, \ldots$ and whenever there is congestion just use the time-steps in between. □

So we can look for α-feasible solutions with as low α and as low makespan.

Algorithm. Each i chooses a random 'start' time $r_i \in \{1, 2, \ldots, \lceil \frac{\beta C}{\log(kD)} \rceil\}$ where $\beta = 3$. Schedule packet i from s_i to t_i starting at r_i and not stopping at all in between.

Lemma 2. *With probability $> 1 - \frac{1}{kD}$, the above schedule is $O(\log(kD))$-feasible.*

Corollary 1. *There exists a feasible schedule with makespan $O(C + D\log(kD))$.*

Proof. The (in)feasible schedule has makespan $\leq \frac{\beta C}{\log(kD)} + D$. The corollary follows from the previous two lemmas. □

Proof. (Proof of Lemma 2) Fix an edge e and a time instant $1 \leq t \leq D + \frac{\beta C}{\log(kD)}$. Let $\mathsf{cong}(e, t)$ be the number of packets passing through edge e at time instant t. We can write $\mathsf{cong}(e, t)$ as a sum of independent indicator variables. Let $X(i, e, t)$ be the indicator that packet i passes through e at time t. Then

$$\mathsf{cong}(e, t) = \sum_{i=1}^{k} X(i, e, t)$$

What is $X(i, e, t)$? If $e \notin p_i$, then $X(i, e, t) = 0$ for all t. Otherwise, let e be the ℓth edge on p_i. Then $X(i, e, t) = 1$ iff $r_i + \ell = t$, that is, $r_i = t - \ell$. In particular, if $\ell < t$ pr if $t - \ell > \frac{\beta C}{\log(kD)}$, then $X(i, e, t) = 0$, otherwise, $X(i, e, t) = 1$ iff $r_i = t - \ell$. In any case, $\mathbf{E}[X(i, e, t)] \leq \frac{\log(kD)}{\beta C}$. And so,

$$\mathbf{E}[\mathsf{cong}(e, t)] \leq \log(kD)/\beta$$

By Chernoff, and using $\beta > 3$,

$$\Pr[\mathsf{cong}(e, t) > 3\log(kD)] \leq 2^{-3\log(kD)} = \frac{1}{k^3 D^3}$$

The number of edges that we need to consider is $\leq kD$. The number of time instants $\leq D + \beta C \leq D + \beta k \leq 3Dk$. So, the probability there exists an edge and time instant t at which more than $3\log(kD)$ packets pass through it is at most $3/(kD)$. □