Lecture 17

Saturday, March 05, 2016
9:46 AM

- Last lecture, showed that approximate sampling from set of independent sets implies approx counting of # of independent sets
- Finish details of analysis
- Analysis extends to other problems with "self-reducibility"
  - See MR 11.3.1 for how to count perfect matchings assuming uniform sampler for perfect matchings
  - <u>Satisfiability</u>: Given SAT instance φ, approximately count # of satisfying assignments.

    For $\alpha \in \{0,1\}^*$, let $N_\alpha = \{\beta \in \{0,1\}^n$ satisfying $\mid \beta_i = \alpha_i$ for $i \in [|\alpha|]\}$. Want to find $|N_\emptyset|$.

    Note that $\max\{|N_{\alpha 0}|, |N_{\alpha 1}|\} \geq \frac{1}{2}|N_\alpha|$.

    Then, $|N_\emptyset| = \frac{|N_\emptyset|}{|N_{\alpha_1}|} \cdot \frac{|N_{\alpha_1}|}{|N_{\alpha_2}|} \cdots \frac{|N_{\alpha_{n-1}}|}{|N_{\alpha_n}|}$

    where $|\alpha_i| = i$ and $\frac{|N_{\alpha_i}|}{|N_{\alpha_{i+1}}|} \geq \frac{1}{2}$. $(N_{\alpha_n} \neq \emptyset)$.

    Estimate each of these ratios by an approximately uniform sampler.
- An art to come up with Markov chains that mix fast.
- We'll look at 2 techniques to analyse mixing time:
  (1) Coupling (2) Canonical paths

<u>Coupling</u>

<u>Def</u>: For dists μ and ν on Ω, $\|\mu - \nu\|_{TV} = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|$.

So, if μ and ν close in TV, then prob of any event occurring wrt μ or ν similar.

Claim: $\|\mu - \nu\|_{TV} = \frac{1}{2}\|\mu - \nu\|_1$ (Check!)

Def: For a MC with transition matrix P, define

$$
\begin{cases}
d(t) = \max_x \| P^t(x,\cdot) - \pi \|_{TV} \\
\bar{d}(t) = \max_{x,y} \| P^t(x,\cdot) - P^t(y,\cdot) \|_{TV} \\
t_{mix}(\varepsilon) = \min \{ t : d(t) \le \varepsilon \} \\
t_{mix} = t_{mix}(1/4).
\end{cases}
$$

Lemma: $t_{mix}(\varepsilon) \le t_{mix} \cdot \log(1/\varepsilon)$.

Def: A coupling between two distributions μ and ν is a pair of random variables (X, Y) such that $X \sim \mu$ and $Y \sim \nu$.

Example: Suppose $\mu = \nu$, uniform on $\{0,1\}$.

Then, (1) (X, Y) where X and Y are independent samples from μ

(2) (X, X) where X is a sample from μ

are both couplings!

Lemma: $\|\mu - \nu\|_{TV} \le \Pr[X \neq Y]$

(In fact, an equality! An "optimal coupling" exists!)

Pf: For any $A \subseteq \Omega$,

$$
\begin{aligned}
\mu(A) - \nu(A) &= \Pr[X \in A] - \Pr[Y \in A] \\
&= \Pr[X \in A] - \Pr[X \in A, Y \in A] \\
&\le \Pr[X \in A] - \Pr[X \in A, Y \in A] \\
&= \Pr[X \in A, Y \notin A] \\
&\le \Pr[X \neq Y]
\end{aligned}
$$

$\therefore \|\mu - \nu\|_{TV} \le \Pr[X \neq Y]$.

close in TV distance,

two distributions ... (X, Y) where $\Pr[X \neq Y]$

- So, to show ... suffices to construct coupling (X, Y) ...
is small.

Def: For a MC, a coupling is a process $(X_t, Y_t)_{t=0}^{\infty}$ such that (X_t) and (Y_t) both separately evolve according to the MC but started at different states.
We assume that once $X_t = Y_t$, $X_{t'} = Y_{t'}$ $\forall t' > t$.
(Simply run the chains together after t).

Key Lemma: Suppose (X_t, Y_t) is a coupling of a Markov Chain with $X_0 = x$ and $Y_0 = y$.

$$\tau_{couple} = \min \{ t : X_t = Y_t \}$$

Then, $\| P^t(x, \cdot) - P^t(y, \cdot) \|_{TV} \leq Pr[\tau_{couple} > t]$

Pf: $\| P^t(x, \cdot) - P^t(y, \cdot) \|_{TV} = Pr[X_t \neq Y_t]$
$= Pr[\tau_{couple} > t]$

Corollary: $d(t) \leq \bar{d}(t) \leq \max_{x,y} Pr[\tau_{couple} > t]$

Example: Lazy random walk on cycle: stay at current vertex w. prob. $\frac{1}{2}$, move left w. prob. $\frac{1}{4}$, move right w. prob. $\frac{1}{4}$.
Coupling: (X_t, Y_t) with $X_0 = x$, $Y_0 = y$. At every t, choose X_t w. prob. $\frac{1}{2}$ and move to left / right with equal prob. Otherwise, choose Y_t and move to left/right with equal prob. Once they meet, they stick together.
Note that they never cross over.
Let $D(t) = |X_t - Y_t|$. Increases or decreases by 1 w. prob. $1/2$ at each step.

$E[\min\{t : D_t = 0 \text{ or } D_t = n\}] \leq n^2$

w. prob. $1/2$ at each step.

Then: $\mathbb{E}[\min\{t : D_t = 0 \text{ or } D_t = n\}] \le n^2$

$$\Pr[\tau_{\text{converge}} > t] \le \frac{n^2}{t}$$

So, $d(t) \le \frac{n^2}{t}$ and $\tau_{\text{mix}} \le 4n^2$.

Example: Lazy random walk on hypercube: stay at current vertex w. prob. $\frac{1}{2}$ and move to a uniform random neighbor otherwise.

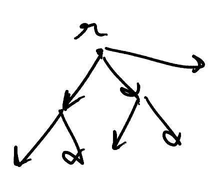
Equiv: Pick a random $i \in [n]$ and set x_i to a random bit.

Coupling: $X_0 = x$, $Y_0 = y$. At every $t \ge 0$, Pick a random $i \in [n]$ and update the i'th bit of both X_t and Y_t to the same random bit to get (X_{t+1}, Y_{t+1})

τ_{couple} is the time when all coordinates have been picked.

Example: Generating random spanning trees.

Consider arborescence rooted at r, all edges directed outwards from r.



Every vertex other than r has exactly one incoming edge.

MC: Choose a random edge (u, r). Add (u, r) to arborescence and remove the only incoming edge to u. Make u the new root.

Coupling: Evolve X_t and Y_t independently, until roots collide. After then, evolve them together.

collide. After then, evolve them together.

$$\mathbb{E}[\tau_{couple}] = \mathbb{E}[\text{time for roots to meet}] + \mathbb{E}[\text{time for trees to become same} \mid \text{same roots}]$$

$$= O(\text{cover time}) = O(n^3).$$

<u>Example</u>: Generating random colorings.

Δ = max degree

<u>Fact</u>: Any graph with max deg Δ can be colored with $\Delta+1$ colors.

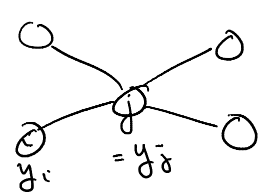
But how to sample a random coloring? Can again be used to approx count # of colorings.

Consider following MC on C-colorings:

- Choose random vertex v
- Color v with random color $c \in [C]$ if permitted. Otherwise, don't change coloring.

<u>Lemma</u>: MC irreducible if $C \geq \Delta + 2$

<u>Pf</u>: Will show that there is a path between any two C-colorings x and y. Order vertices in some order and in that order, change x_i to y_i. May not be possible if blocked by future vertex j but can recolor if $C \geq \Delta + 2$.



<u>Lemma</u>: Stationary distribution is uniform. (Exercise!)

<u>Lemma</u>: Mixing time is $O(n \log n)$ if $C > 4\Delta + 1$.

: both MC copies

Lemma: Mixing time is $O(n \log$

Pf: Consider trivial coupling: both MC copies choose the same vertex and same color.

$$\Pr[d_{t+1} = d_t - 1 \mid d_t > 0] \geq \frac{d_t}{n} \cdot \frac{c - 2\Delta}{c}$$

$$\Pr[d_{t+1} = d_t + 1 \mid d_t > 0] \leq \frac{\Delta d_t}{n} \cdot \frac{2}{c}$$

$$\Rightarrow \mathbb{E}[d_{t+1} \mid d_t] \leq d_t \cdot \left(1 - \frac{c - 4\Delta}{cn}\right)$$

$$\Rightarrow \mathbb{E}[d_{t+1}] \leq \mathbb{E}[d_t] \cdot \left(1 - \frac{c - 4\Delta}{cn}\right)$$

$$\leq \mathbb{E}[d_t] \cdot \left(1 - \frac{1}{cn}\right)$$

$$\Rightarrow \Pr[d_t \geq 1] \leq n e^{-t/cn} \quad \text{since } d_0 \leq n.$$