Lecture 18

Wednesday, March 09, 2016
11:52 AM

Agenda: (1) Finish examples of coupling
(2) Conductance and canonical paths

Conductance

- Coupling method directly bounds TV distance
- But two problems with it:
  (1) Requires cleverness to come up with coupling with bounded coupling time
  (2) Provably in some cases, optimal coupling is "non-causal", so very unnatural.
- Conductance bounds l_2-distance which in turn bounds l_1-distance
- $\|\mu - \nu\|_2 = \sqrt{\sum_{x \in \Omega} (\mu(x) - \nu(x))^2}$
- $d_2(t) = \max_x \| P^t(x, \cdot) - \pi \|_2$
- **Fact**: $\frac{\|\mu-\nu\|_1}{\sqrt{n}} \leq \|\mu - \nu\|_2 \leq \|\mu - \nu\|_1$. So, for mixing, need to get $d_2(t)$ down to $O(1/\sqrt{n})$

Spectral theory

- For simplicity, consider MC with symmetric transition matrix and uniform stationary distribution.
- P has orthogonal basis of eigenvectors $v_1, \ldots, v_n$ with eigenvals $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.
- $x P^t = (c_1 v_1 + \cdots + c_n v_n) P^t$
  $= c_1 \lambda_1^t v_1 + c_2 \lambda_2^t v_2 + \cdots + c_n \lambda_n^t v_t$
- $\lambda_1 = 1$ and $v_1 = \pi$
- Assume chain is aperiodic, so, $\lambda_n > -1$
- Then, if $|\lambda_2| > |\lambda_n|$, λ_2 controls decay.
- This is true for lazy walks that have self-loops w. prob. $\frac{1}{2}$.

  $P' = \frac{1}{2}(I + P)$

  $\Rightarrow \lambda_{P'} = \frac{1}{2}(1 + \lambda_P) \geq 0.$

- $\| x P^t - \pi \|_2^2 = \| \sum_{i=2}^{n} c_i \lambda_i^t v_i \|_2^2$

$\Rightarrow \lambda_{P'} = \frac{1}{2}(1+\lambda_P) \geq 0$

- $\| x P^t - \pi \|_2^2 = \| \sum_{i=2}^{n} c_i \lambda_i^t v_i \|_2^2$

$$= \sum_{i=2}^{n} \lambda_i^{2t} c_i^2 \leq \sum_{i=2}^{n} \lambda_2^{2t} c_i^2$$

$$= \lambda_2^{2t} \cdot \| x - \pi \|_2^2$$

$$\Rightarrow d_2(t) \leq \lambda_2^t \cdot \| x - \pi \|_2$$

$$\leq 2 \cdot \lambda_2^t$$

$$\Rightarrow d(t) \leq \lambda_2^t \cdot \sqrt{n} \Rightarrow t_{mix}(\varepsilon) \leq \frac{\ln(\sqrt{n}/\varepsilon)}{\ln(1/\lambda_2)}$$

$$\leq \frac{\ln(\sqrt{n}/\varepsilon)}{1 - \lambda_2}$$

- Thus, need to show lower bound on <u>spectral gap</u> : $1 - \lambda_2$.
- If transition matrix P is asymmetric, then consider

$$\tilde{P}_{i,j} = \sqrt{\frac{\pi_j}{\pi_i}}\, P_{i,j} \leftarrow \text{symmetric if chain is "reversible"}.$$

Conductance

- λ_2 not so easy to calculate.
- But there exists a powerful connection between λ_2 and graph expansion.
- For $S \subseteq \Omega$, $\Phi(S) = \dfrac{\sum_{i \in S, j \notin S} \pi_i P_{ij}}{\sum_{i \in S} \pi_i}$.

If π is uniform, $\Phi(S) = \dfrac{\sum_{i \in S, j \notin S} P_{ij}}{|S|}$. $\Phi = \min_{\pi(S) \leq \frac{1}{2}} \Phi(S)$

- <u>Cheeger's Inequality</u> : $\dfrac{\Phi^2}{2} \leq 1 - \lambda_2 \leq 2\Phi$

- So, $t_{mix}(\varepsilon) \leq \dfrac{2 \ln(\sqrt{n}/\varepsilon)}{\Phi^2}$

- <u>Example</u>: Lazy random walk on cycle

$$\Phi(S) \geq \frac{\frac{1}{2}}{|S|} = \frac{1}{2|S|} \Rightarrow \Phi \geq \frac{1}{n}.$$

$$\Rightarrow t_{mix} = O(n^2 \log n).$$

~~Canonical Paths~~

- Can be hard to guess which subset gives the minimum conductance
- For each pair of states x and y, assign a canonical path $\gamma_{x,y}$, such that there are at most bN paths passing through any single edge.
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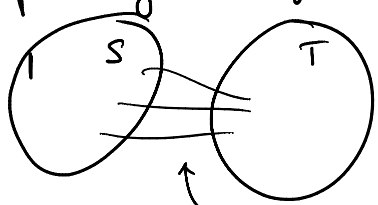

$\geq |S| \cdot |T|$ paths

$\Rightarrow \geq \frac{|S| \cdot |T|}{b} \geq |S|$

$\Rightarrow$ Lower bound on conductance

<u>Example</u>: Lazy random walk on hypercube

- Canonical path from x to y by bit-fixing
- If (u,v) is an edge with $u_k \neq v_k$, then γ_{xy} passes thru (u,v) if $u_k, \ldots, u_n = x_k, \ldots, x_n$ and $v_1, \ldots, v_k = y_1, \ldots, y_k$ $\Longrightarrow 2^{n-1}$ paths passing thru (u,v) $\Rightarrow b = \frac{1}{2}$.

$$\text{Conductance} \geq \frac{\frac{|S|}{2 \cdot \frac{1}{2}} \cdot \frac{1}{2n}}{|S|} = \frac{1}{2n}$$

$\Rightarrow \tau_{min} = O(n^2 \lg 2^n) \leftarrow$ Coupling gave bound of $O(n \log n)$

$= O(n^3)$

<u>Example</u>: Uniformly sampling matchings

- Can be used to approximately count # matchings

- MC has states which are matchings
- Transitions from current matching M:

- Pick an edge e uniformly
- With prob. $1/2$, $M \leftarrow M \setminus \{e\}$.
- O.w., $M \leftarrow M \cup \{e\}$ if possible. If not, stay at M.

- Undirected
- Connected
- Non-bipartite
- Regular with deg n
- Define canonical path between M_1 and M_2:
    - $M_1 \oplus M_2$ has deg ≤ 2, so consists of paths and cycles. Moreover, consecutive edges alternate between being in M_1 and M_2.
    - Order the components of $M_1 \oplus M_2$ in some canonical way (e.g. id of smallest vertex in each component), a starting vertex in each component (e.g. smaller endpoint if comp is a path, smallest vertex if comp is a cycle), and orientation (away from starting vertex if comp is a path, and outward from starting vertex using edge in M_1 if comp is a cycle)
    - Suppose comp is a path: $e_1, \ldots, e_k$. Suppose e_1 is to be added. Starting from $i=1$, delete e_{i+1}, add e_i and $i \leftarrow i+2$.
    - Suppose comp is a cycle: $e_1, \ldots, e_k$. We said $e_1 \in M_1$, so has to be deleted. Delete e_1. Then starting from $i=2$, delete e_{i+1}, add e_i, and $i \leftarrow i+2$.
    - ~~Claim~~: Suppose we are given $M_1 \oplus M_2$ and matchings M_a and M_b, where $M_a \to M_b$ is an intermediate transition in above canonical path from M_1 to M_2. Then, M_1 and M_2 can be recovered.
    - Pf: Let e be the edge modified in $M_a \to M_b$. If e is added, [...] M_2; else, e in M_1. The other edges in the component [...] $M_a \to M_b$ as the edges

of $M_1 \oplus M_2$ can be determined to be in M_1 or M_2 ...
alternate. To obtain rest of M_1, take XOR of M_a with comp's of $M_1 \oplus M_2$ before e's comp. Rest of M_2 is XOR of M_b with comp's of $M_1 \oplus M_2$ after e's comp.

- Claim: $(M_1 \oplus M_2) - M_a$ consists of a matching plus ≤ 2 edges, if $M_a \to M_b$ is a transition in the canonical path.
- Pf: Easy to check. Worst case is if there's a cycle of length 3 in $(M_1 \oplus M_2) - M_a$ and there can be a path of length 3 surrounding an edge of M_2 because both edges of M_1 may be deleted in M_a.

- So, M_1 and M_2 may be recovered from M_a and M_b if an $\underbrace{\text{additional matching plus} \leq 2 \text{ edges}}_{N m^2 \text{ possibilities}}$ specified

$$\Rightarrow b = m^2$$

$$\Rightarrow \text{Conductance} \geq \frac{\frac{|S|}{2m^2} \cdot \frac{1}{2m}}{|S|} = \frac{1}{4m^3}$$

$$\Rightarrow t_{mix} \leq O\left(\frac{1}{\Phi^2} \ln 2^m\right) = O(m^7)$$