Lecture 19

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2:26 AM

Martingales

Def: A sequence $Z_0, Z_1, \ldots$ is a martingale with respect to a sequence $X_0, X_1, \ldots$ if:

(1) Z_i is a function of $X_0, \ldots, X_i$; $\forall i$

(2) $\mathbb{E}[|Z_i|] < \infty$

(3) $\mathbb{E}[Z_{i+1} \mid X_0, \ldots, X_i] = Z_i$

Ex: A gambler makes a series of fair bets, of varying amounts perhaps depending on each other. Let X_i be his winnings in the i'th game, and let Z_i be his cumulative winnings in $1, \ldots, i$ games. Then, if the amounts he bets is finite, Z's form a martingale w.r.t. X's.

Stopping Times

For any fixed i,

$$\mathbb{E}[Z_i] = \mathbb{E}[\mathbb{E}[Z_i \mid X_0, \ldots, X_{i-1}]] = \mathbb{E}[Z_{i-1}] = \cdots = \mathbb{E}[Z_0]$$

But it may sometimes be interesting to know $\mathbb{E}[Z_i]$ when i depends on $X_0, \ldots, X_i$..

Ex: Suppose τ be the first time that the gambler's winnings reaches ₹100. Clearly, $\mathbb{E}[Z_\tau] = 100 \neq \mathbb{E}[Z_0]$.

Ex: Suppose τ' be the first time that the gambler's winnings reaches 100 or -50. Below theorems shows $\mathbb{E}[Z_{\tau'}] = \mathbb{E}[Z_0] = 0$.

Def: A non-negative, random variable T is called a ~~stopping~~

Def. A ... ... $\}$ if the event $T = n$ only dep...
time for $\{Z_0, Z_1, \dots\}$
$Z_0, Z_1, \dots, Z_n$.

Martingale Stopping Thm: If $\{Z_i : i \geq 0\}$ is a martingale w.r.t. $\{X_i : i \geq 0\}$, and T is a stopping time, then $\mathbb{E}[Z_T] = \mathbb{E}[Z_0]$ if either of following:

(1) The Z_i's are bounded by constant
(2) T is bounded by constant
(3) $\mathbb{E}[T] < \infty$ and $\mathbb{E}[|Z_{i+1} - Z_i| \mid X_1, \dots, X_i]$ is bounded by constant

Ex (contd.): $-50 \leq Z_i \leq 100$, so theorem applies.

$$\Pr[\text{gambler loses } 50 \text{ before winning } 100] = \frac{100}{100+50} = \frac{2}{3}.$$

even for dependent bets!

Ex: (Wald's Theorem)
Let $X_1, \dots,$ be non-negative, i.i.d. random variables, and let T be a stopping time for this sequence. Suppose $\mathbb{E}[T]$, $\mathbb{E}[X]$ bounded.
Then: $$\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbb{E}[X_i] \cdot \mathbb{E}[T]$$

Pf: Let $$Z_i = \sum_{j=0}^{i} (X_j - \mathbb{E}[X])$$

Clearly, a martingale. Then, since $\mathbb{E}[|Z_{i+1} - Z_i|] \leq 2 \cdot \mathbb{E}[X_i]$ bounded, it follows that

$$\mathbb{E}[Z_T] = 0$$

or, $$\mathbb{E}\left[\sum_{j=1}^{T} X_j\right] = \mathbb{E}\left[\sum_{j=1}^{T} \mathbb{E}[X]\right]$$

$$= \mathbb{E}[T] \cdot \mathbb{E}[X].$$

Ex: Waiting times for patterns in coin tosses

Sequence of coin tosses: THHT.... Call the sequence x

Fix a pattern string $p = THT$.

Let τ be the first time at which the pattern has occurred.

Consider a gambler G_i. G_i bets 1 that $x_i = p_1$. If he wins, he wins 2 and bets that on $x_{i+1} = p_2$. If he wins, he gets 4 and bets that on $x_{i+2} = p_3$. If he wins now, he gets 8.

- Sum of martingales still a martingale
- Z_i = total payoffs to gamblers till i'th toss.
- $\mathbb{E}[Z_\tau] = 0$

$$\Rightarrow \mathbb{E}\left[-(\tau-2) + 7 + 1\right] = 0$$

$$\Rightarrow \mathbb{E}[\tau] = 10. \qquad \Rightarrow \text{Generalize?}$$

~~Azuma's inequality~~

Thm: Let $X_0, X_1, \ldots, \ldots$ be a martingale with

$$|X_k - X_{k-1}| \leq c_k$$

Then, $\forall\, t \geq 0$ and $\lambda > 0$,

$$\Pr\left[\,|X_t - X_0| \geq \lambda\right] \leq 2e^{-\lambda^2/2\left(\sum_{i=1}^{k} c_i^2\right)}$$

Cor: If as above, and $|X_k - X_{k-1}| \leq c$, then

$$\Pr\left[\,|X_t - X_0| \geq \lambda c\sqrt{t}\right] \leq 2e^{-\lambda^2/2}$$

Rem: Generalizes (qualitatively) Chernoff bound for Bernoulli variables. Let Z_i be independent Bernoulli's. $S = \sum_{i=1}^{n} Z_i$. Define $X_i = \mathbb{E}[S \mid Z_1, \dots, Z_i]$. This is a martingale w.r.t. Z_i's as

$$\mathbb{E}[X_{i+1} \mid Z_1, \dots, Z_i]$$
$$= \mathbb{E}\big[\mathbb{E}[S \mid Z_1, \dots, Z_{i+1}] \mid Z_1, \dots, Z_i\big]$$
$$= \mathbb{E}[S \mid Z_1, \dots, Z_i] = X_i$$

Pf Sketch: Similar to the proof of Chernoff bound.

$$\Pr[X_t - X_0 \geq \lambda] = \Pr\left[e^{\alpha(X_t - X_0)} \geq e^{\alpha\lambda}\right]$$

Define $Y_i = X_i - X_{i-1}$. Note $\mathbb{E}[Y_i \mid X_{i-1}] = 0$.

Claim: If Z is a random var such that $\mathbb{E}[Z] = 0$ and $a \leq Z \leq b$, then $\mathbb{E}[e^{\alpha Z}] \leq e^{\alpha^2 (b-a)^2/8}$.

So, $$\mathbb{E}\left[e^{\alpha(X_t - X_0)}\right] = \mathbb{E}\left[e^{\sum_{i=1}^{t} \alpha Y_i}\right]$$
$$= \mathbb{E}\left[e^{\sum_{i=1}^{t-1} \alpha Y_i}\right] \cdot \mathbb{E}\left[e^{\alpha Y_t} \mid X_0, X_1, \dots, X_t\right]$$
$$\leq \mathbb{E}\left[e^{\sum_{i=1}^{t-1} \alpha Y_i}\right] \cdot e^{\alpha^2 c_t^2/2}$$
$$\leq e^{\frac{\alpha^2}{2} \sum_{i=1}^{t} c_i^2}$$

$$\Pr[X_t - X_0 \geq \lambda] \leq \min_{\alpha > 0} \frac{e^{\lambda\alpha}}{e^{\frac{\alpha^2}{2} \sum c_i^2}} \leq e^{-\frac{\lambda^2}{2 \sum c_i^2}} \qquad \square$$

Generalized Azuma for Doob Martingales:

Generalized Azuma for Doob Martingales:

Suppose $f(x_1, \dots, x_n)$ is a function s.t. $\forall i$,
$|f(x) - f(x')| \le d_i$ if x and x' only differ in i'th coordinate.

Then if $X_1, \dots, X_n$ are independent random vars,

$$\Pr\left[\,|f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)]| > \lambda\right] \le 2\exp\left(-\frac{2\lambda^2}{\sum d_i^2}\right)$$

Examples

(1) Contiguous patterns

(2) Stochastic bin packing: Given n items of sizes in $[0,1]$, pack them into as few unit size bins as possible. Suppose item sizes are indep chosen in $[0,1]$. Then, use Azuma on Doob martingale for $B(x_1, \dots, x_n)$ = min number of bins.

Clearly, $|\mathbb{E}[B \mid X_1, \dots, X_i] - \mathbb{E}[B \mid X_1, \dots, X_{i-1}]| \le 1$

Also, B has Lipschitz constant 1.

So, $\Pr[\,|B - \mathbb{E}B| > \lambda] \le 2e^{-2\lambda^2/n}$

(3) Balls and bins (n balls, m bins, randomly thrown)

Z = # of empty bins

$Z_i = 1[\text{bin } i \text{ is empty}]$

$Z = \sum Z_i$ but Z_i's not indep

But Z again satisfies Lipschitz condition with constant 1.

$$\Rightarrow \Pr[|Z - \mathbb{E}Z| > \lambda] \leq 2 \cdot e^{-2\lambda^2/n}$$

(4) Chromatic number

Vertex exposure martingale:

$$Y_i = \mathbb{E}[\chi(G) \mid G_1, \dots, G_i]$$

where G_i is induced subgraph on vertices $\{1, \dots, i\}$.

$$Y_{i+1} \leq Y_i + 1$$

$$\Rightarrow \Pr[|\chi(G) - \mathbb{E}\chi(G)| > \lambda] \leq e^{-2\lambda^2/n}$$