# A Crash Course in Linear Programming

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## A CRASH-COURSE IN LINEAR PROGRAMMING

**A general LP:** $\min \ c^T x \ : \ Ax \geq b$

$x \in \mathbb{R}^n$ : variables

$A \in \mathbb{R}^{m \times n}$ : Constraint matrix, (usually $m \geq n$)

$c$ : obj-function, $c \in \mathbb{R}^n$

$$c^T x = c \cdot x = \langle c, x \rangle = \sum_{i=1}^{n} c_i x_i .$$

**Picture when $n=2$:**

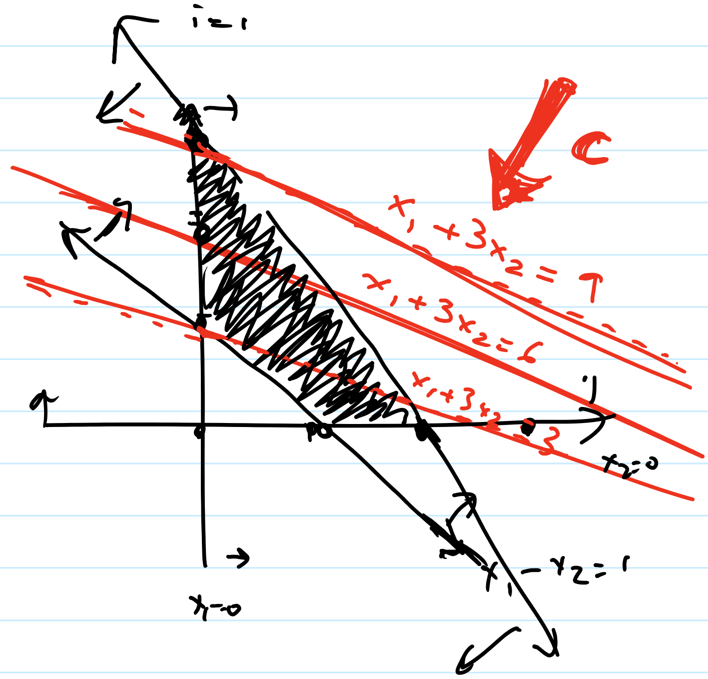
$\min \ x_1 + 3x_2$

$3x_1 + 2x_2 \geq 6$

$x_1 - x_2 \geq 1$

$x_1 \geq 0$

$x_2 \geq 0$




**Linear Algebra Preliminaries**

- Given $A$, $\{a_1, \ldots, a_m\} \subseteq \mathbb{R}^n$ are the $m$-rows

  $\{A_1, \ldots, A_n\} \subseteq \mathbb{R}^m$ — " — $n$-cols.

- $\text{Span}(v_1, \ldots, v_k) := \left\{ \sum_{i=1}^{k} \lambda_i v_i \ : \ \lambda_i \in \mathbb{R} \right\}$

  This is an example of a Vector Space.

  - if $v \in V \Rightarrow \alpha v \in V$
  - $u, v \in V \Rightarrow u + v \in V$

- Lin. Ind: A set $\{v_1, \ldots, v_k\}$ of vectors are lin. independent iff

$$\sum_{i=1}^{k} \lambda_i v_i = \vec{0} \iff \lambda_i = 0 \ \forall i = 1..k$$

eg: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- FACT :- Any maximal collection of lin. ind. sets in a vector space has the same cardinality.

The size of this "basis" is the dim (V).

Try to prove this.

- Given a matrix $A \in \mathbb{R}^{m \times n}$, two important V-spaces

① Row-Space $\mathcal{R}$ $\equiv \text{Span}\{a_1, \ldots, a_m\} \subseteq \mathbb{R}^n$

② Col-Space $\mathcal{C}$ $\equiv \text{Span}\{A_1, \ldots, A_n\} \subseteq \mathbb{R}^m$

row-rank $\equiv \dim(\mathcal{R})$ ; col-rank $\equiv \dim(\mathcal{C})$

$\downarrow$ max # of lin-ind rows ; $\downarrow$ max # of lin-ind cols

- AMAZING FACT: row-rank$(A)$ = col-rank$(A)$
$\equiv$ rank$(A)$

- Ways to think of $Ax \doteq \sum_{j=1}^{n} A_j x_j$ ie. a linear comb of cols.

$\therefore \quad \underset{\forall x \in \mathbb{R}^n}{Ax \in \mathcal{C}}, \quad \underset{\forall y \in \mathbb{R}^m.}{y^T A \in \mathcal{R}}$

Proof:

- In $\mathbb{R}^n$ (which is an inner-prod-space, ie, it has an inner product defined on it) any V-space $\subseteq \mathbb{R}^n$, has a "perpendicular" V-space $V^\perp$ also.... $V^\perp = \{u : \langle u, v \rangle = 0 \ \forall v \in V\}$

Fact: $\dim(V) + \dim(V^\perp) = n$

$$\mathcal{R}^\perp \equiv \{ \underset{\in \mathbb{R}^n}{x} : (y^T A)x = 0, \forall y \in \mathbb{R}^m \}$$

$$\equiv \{ \underset{\in \mathbb{R}^n}{x} : Ax = 0 \}$$

Let $\{r_1, \ldots, r_d\}$ be a basis of $\mathcal{R}^\perp \subseteq \mathbb{R}^n$

note $d = n -$ row-rank$(A)$

This can be completed to a basis of $\mathbb{R}^n$

$$B = \{r_1, \dots r_d, s_{d+1}, \dots, s_n\}$$

$$\therefore \forall v \in \mathbb{R}^n, \qquad v = \sum_{i=1}^{d} \lambda_i r_i + \sum_{i=d+1}^{n} s_i \beta_i$$

$$\therefore \mathcal{C} \equiv \{Av \mid v \in \mathbb{R}^n\}$$

$$\equiv \{A(\sum \lambda_i r_i + \sum \beta_i s_i) \mid \lambda_1, \dots, \lambda_d, \beta_{d+1} \dots, \beta_n\}$$

$$= \{A \cdot \sum \beta_i s_i \mid \beta_{d+1} \dots \beta_n\}$$

$$\because Ar_i = 0 \ \forall i$$

$$\therefore \underbrace{\dim(\mathcal{C})}_{\text{Col-rank}} = \underbrace{n-d}_{\text{row-rank}}$$

$\blacksquare$

- A matrix $A$ is said to have full row-rank if row-rank$(A) = n$.

$$\implies \mathcal{R}^{\perp} \equiv \{\vec{0}\}$$

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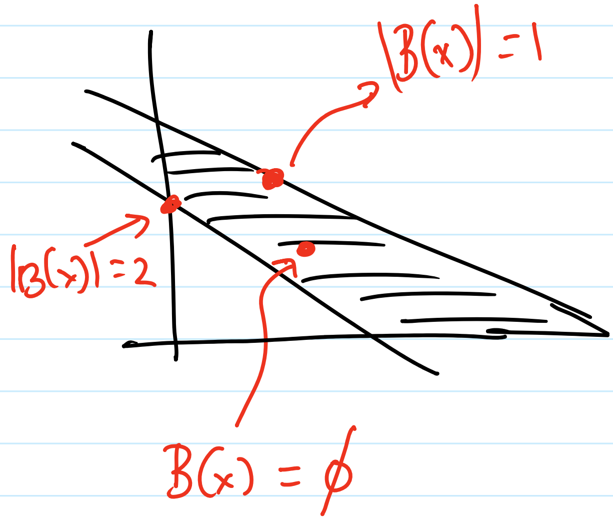
Coming back to LP's ....

- $F := \{x \mid Ax \geq b\}$ is called the feasible region

- Given $x \in F$, let $B(x) \subseteq \{a_1, \dots, a_m\}$ be the set of inequalities that hold with equality.

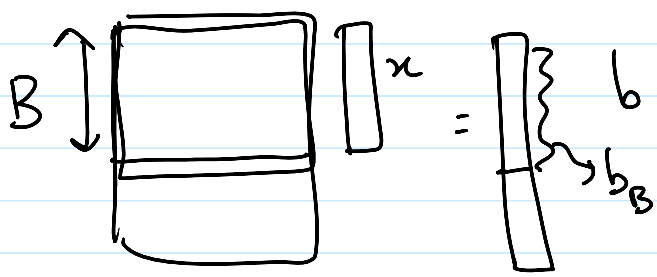
- Henceforth we assume $A$ has full row-rank.




**BASIC FEASIBLE SOLN:** Any $x \in F$ is a basic feasible soln if $B(x)$ forms a basis of $\mathbb{R}^n$.

Also called an EXTREME POINT SOLN or a VERTEX solution.

All bases of $\{a_1, \dots, a_m\}$ $\longleftrightarrow$ Basic Feasible Solutions.




$x = B^{-1} b_B$

**Thm:** Any LP has an Optimum Solution @ a basic feasible soln.

Pf :- If $x$ is an opt-soln & $B(x) \neq$ full-row-rnk, then $\exists\, v \in \mathbb{R}^n$ st $v^T a_i = 0 \ \forall\, a_i \in B(x)$

Consider $x + \delta v$ . . . . finish the proof

## LOCAL-OPT = GLOBAL-OPT

- Fact: For any basis $B$ (of any v-space), $\forall i \in B$, $\exists\, i' \notin B$ s.t. $B - i + i'$ is also a basis.

- $\text{cost}(B) := c^T x_B = c^T B^{-1} b_B$

- Let $B^*$ be the "local-opt" basis.

  i.e. $\forall i \in B^*$, $\forall j \notin B^*$ if

  $B_i = B^* - i + j$ is a basis,

  then $\text{cost}(B_i) \geq \text{cost}(B^*)$

Thm: $x_{B^*}$ is a Global OPT.

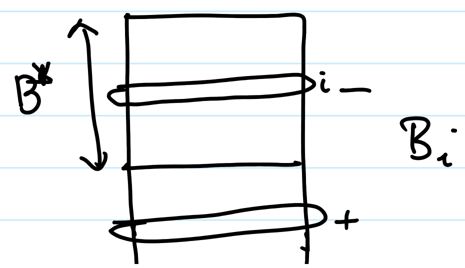
Pf: $\hat{x} := x_{B^*} = {B^*}^{-1} b_{B^*}$

Geometry of BFS

& $x_i := x_{B_i}$, $i \in B^*$

What is . .

$$\therefore\ A(x_i - \hat{x})\Big|_{B^*}.$$




⋮

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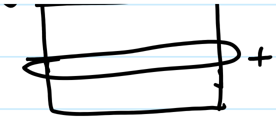


⋮

* it has the $i^{th}$ coor $> 0$
and rest all $0$

$$\therefore \quad A(x_i - \tilde{x})\big|_{B^*} = \gamma_i e_i \quad \text{for some } \gamma_i > 0$$

$$i = 1 \dots n$$

$\therefore$ ① $(x_i - \tilde{x})$'s span row-span$(A) = \mathbb{R}^n$

$\Rightarrow$ ② Any $x \in F$ must satisfy

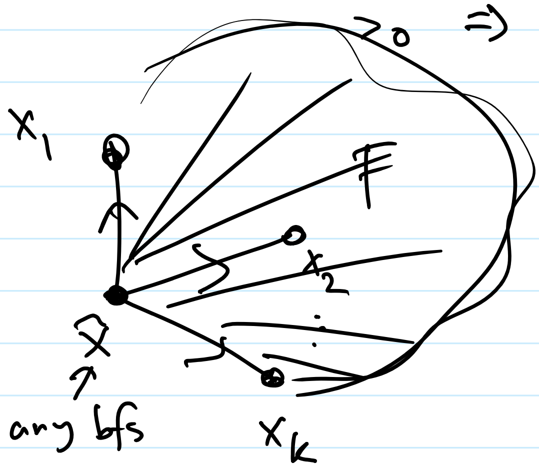
$$(x - \tilde{x}) = \sum \alpha_i (x_i - \tilde{x})$$

with $\boxed{\alpha_i \geq 0}$

Why? Again multiply by $A$ and restrict attn to $B^*$ coor

$$0 \leq A(x - \tilde{x})\big|_{B^*} = \sum \alpha_i \underset{> 0}{\underset{\uparrow}{\gamma_i}} e_i \quad \Rightarrow \quad \alpha_i \geq 0$$

Picture:




Now we are done. If $\hat{x}$ is LOCAL OPT, then, $c^T(x_i - \hat{x}) \geq 0$

if $x^*$ is GLOBAL OPT

$$0 \geq c^T(x^* - \hat{x}) = c^T\Big(\sum \underset{\substack{\uparrow \\ \geq 0}}{\alpha_i}(x_i - \hat{x})\Big)$$

$$\geq 0$$

$\Longrightarrow$ we have $=$ everywhere

$\Longrightarrow$ $c^T x^* = c^T x$ ∎

This is the "Simplex" method

... ok, not quite

... SILLY SIMPLEX METHOD