Lecture 11

Saturday, April 29, 2017 5:34 PM

The Dual Linear Program

- $\min\limits_{\underbrace{Ax \geq b}} c^T x =: p^*$

  $\forall i = 1, \dots, m : \ a_i^T x \geq b_i$

- Moving constraints to the objective

  – Introduce a "penalty" $y_i \geq 0$ for every constraint $i = 1 \dots m$.

– Consider this "Lagrangean" of the problem

$$\underset{y \geq 0}{L(x,y)} = c^T x + \sum_{i=1}^{m} y_i \underbrace{(b_i - a_i^T x)}_{\substack{+ve \text{ if } b_i > a_i^T x \\ \text{ie } x \text{ isn't feas} \\ -ve \text{ o/w}}}$$

– Define :- $h(y) = \min\limits_{x \in \mathbb{R}^n} L(x,y)$

Claim :- For any $y \geq 0$, $h(y) \leq p^*$

Pf :- If x^* achieves optimum, we get

$$h(y) \leq c^T x^* + \sum_{i=1}^{m} y_i^T (b_i - a_i^T x)$$

$$\leq c^T x^* = p^*$$

Since $h(y) \leq p^*$, for all $y \geq 0 \dots$

$$\boxed{\max h(y) \leq p^*}$$

$$\boxed{\max_{y \geq 0} h(y) \leq p^*}$$

- Let us look at what $h(y)$ looks like.

$$h(y) = \min_{x \in \mathbb{R}^n} L(x, y)$$

$$= \min_{x \in \mathbb{R}^n} \left\{ c^T x + \sum_{i=1}^{m} y_i (b_i - a_i^T x) \right\}$$

$$= \sum_{i=1}^{m} b_i y_i + \underbrace{\min_{x \in \mathbb{R}^n} \left\{ \left(\sum_{i=1}^{m} y_i a_i - c \right)^T x \right\}}_{= \begin{cases} 0 & \text{if } y^T A = c \\ -\infty & \text{o/w} \end{cases}}$$

$$\therefore \quad h(y) = \begin{cases} b^T y & \text{if } y^T A = c \\ -\infty & \text{o/w} \end{cases}$$

$$\therefore \quad d^* = \boxed{\begin{array}{c} \max \ b^T y \\ y^T A = c \\ y \geq 0 \end{array}} \leq p^*$$

WEAK DUALITY

THE DUAL LP.

Mechanical Process:

$$\min\ c^T x \qquad\qquad \max\ b^T y$$
$$Ax \geq b \qquad\qquad y^T A = c$$
$$\qquad\qquad\qquad y \geq 0$$

ineq

x is free $\Leftarrow$

ineq

$$\min\ c^T x \qquad\qquad \max\ b^T y$$
$$Ax = b \qquad\qquad y^T A \leq c$$
$$x \geq 0 \qquad\qquad y \text{ free}$$

Example

Set-Cover LP:
- Sets : $S_1, \ldots, S_m$
- elts : $1, 2, \ldots, n$

$$\min \sum_{i=1}^{m} c_i x_i \qquad\qquad \max \sum_{j=1}^{n} y_j$$

$$\forall j = 1 \ldots n : \sum_{i : j \in S_i} x_i \geq 1 \qquad\qquad \forall i : \sum_{j \in S_i} y_j \leq c_i$$

$$x \geq 0 \qquad\qquad y \geq 0$$

Total mass in any set is $\leq$ Cost of set

Variable per elt.

ANY FEASIBLE DUAL GIVES A LOWER BOUND ON

A LOWER BOUND ON OPT

$$OPT \geq p^* \geq d^* \geq \text{feas-dual.}$$

APPLICATION

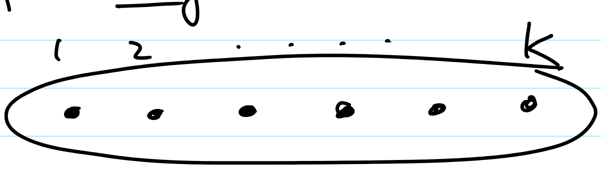
- Recall the GREEDY Alg for Set-Cover
- When element j is covered $\overbrace{\text{for the 1st time}}$ by a set S_i, assign

$$\tilde{y}_j = \frac{c(S_i)}{|S_i \cap X_i|}$$

← # of new elts that is covered by S_i

- $$\sum \tilde{y}_j = ALG$$

- Now fix <u>any</u> set S



Order elts in order which covered by algo.

$$\tilde{y}_1 = \frac{c(S_i)}{|S_i \cap X_i|} \leq \frac{c(S)}{k}$$

$$\vdots$$

$$\tilde{y}_j \leq \frac{c(S)}{k-j+1}$$

J $\qquad\qquad\qquad$ $k-j+1$

$$\Rightarrow \forall S: \sum_{j \in S} \tilde{y}_j \le c(S) \cdot H_k$$

$\therefore$ y defined as $y_j = \frac{1}{H_\Delta} \cdot \tilde{y}_j$ is a feasible dual with $\sum_{j=1}^{n} y_j = \frac{ALG}{H_\Delta}$

$$\therefore \; OPT \geq LP \geq \frac{ALG}{H_\Delta}$$

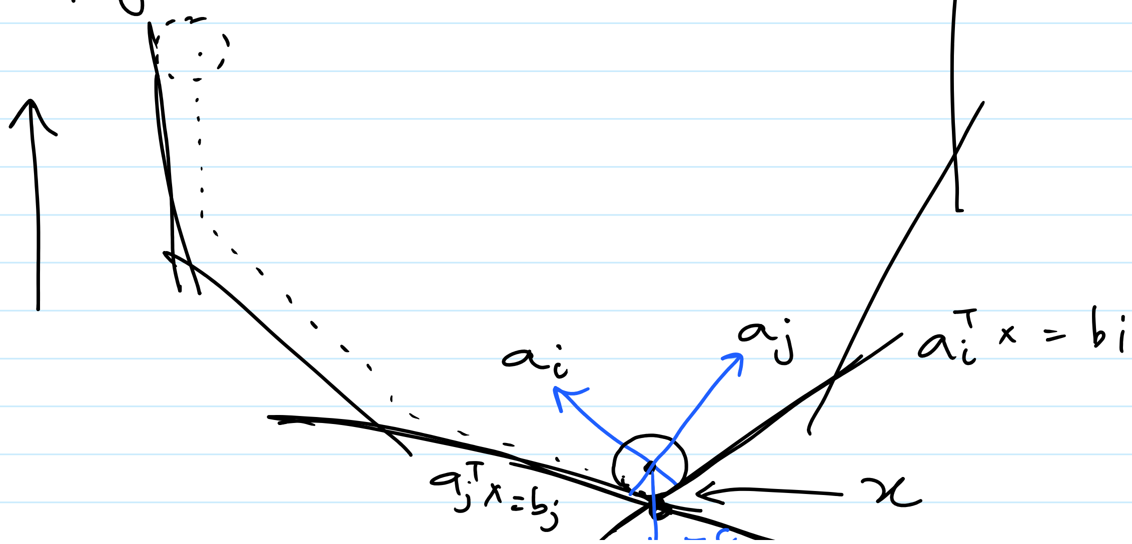
∎

Strong Duality Theorem

<u>Thm</u>: $p^* = d^*$ $\qquad$! AMAZING THEOREM

<u>"Proof"</u>:

A "physics" intuition



- Align the polytope $\{Ax \geq b\}$ s.t. $\vec{c}$ is pointing "upwards"

- Now, imagine a marble falling in this jaggedy bowl.
- It will come to rest @ the pt x which minimizes $c^T x$.
- The "force" $-c$ is balanced by the "normal" fources on the surfaces that marble is touching.

- So if $\{\vec{a}_1, \dots, \vec{a}_n\}$ are the normals of the walls that define the nadir of this bowl, we have

$$(*) \qquad \sum_{i=1}^{n} y_i \vec{a}_i = \vec{c} \qquad \text{with } y_i \geq 0$$

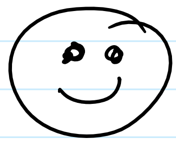
- Complete it to $y \in \mathbb{R}^m$ by defining $y_i = 0$ for all the "other walls"

- Multiplying by x on both sides

$$\sum^{n} \quad {}^{T} \qquad\qquad {}^{T} \qquad\qquad *$$

$$\sum_{i=1}^{n} y_i a_i^T x = c^T x = p^*$$

$$\sum_{i=1}^{n} y_i b_i \overset{\text{||}}{\leq} d^*$$



Converting the physics into a math proof:

Proof:-

For simplicity, we assume $P = \{Ax \geq b\}$ is non-degenerate. That is, for any bfs x, the set $B(x)$ the set of constraints sat. with equality and is full dimn by defn is of size exactly n.

Let x be the optimum solution. Let $B = \{a_1, \ldots, a_n\}$ be the set of constraints satisfied with equality.

Since B is a basis of $\mathbb{R}^n$, there is a unique representation of the vector $c \in \mathbb{R}^n$

$$c = \sum_{i=1}^{n} y_i a_i$$

$$c = \sum_{i=1} y_i a_i$$

Claim :- $y_i \geq 0$, $\forall i = 1 \ldots n$

Once this claim is proven, the "physics" has been made formal.

$$d^* \geq \sum_{i=1}^{n} b_i y_i = \sum_{i=1}^{n} y_i a_i^T x = c^T x = p^* \geq d^*$$

$\because$ y is feasible dual

Pf :- Suppose not, and by renaming say $y_1 < 0$.

- Choose $v \in \mathbb{R}^n$ s.t.
  - (a) $v \perp \{a_2, \cdots, a_n\}$
  - (b) $v^T a_1 > 0$

eg :- We can pick

$$v = a_1 - \text{proj}\left(a_1, \text{span}\{a_2, \ldots, a_n\}\right)$$

- $\tilde{x} = x + \delta v$ for some small $\delta > 0$ which we will fix soon.

- Since $a_j^T x > b_j \quad \forall j \notin \{1, \cdots, n\}$

define $$\delta := \min_{j \notin [n]} \frac{(a_j^T x - b_j)}{|a_j^T v|} > 0$$

- We claim $\tilde{x}$ is feasible

① $a_1^T \tilde{x} = a_1^T x + \delta a_1^T v > b_1$

② $a_i^T \tilde{x} = a_i^T x + \delta a_i^T v = b_i$

$i = 2, \dots, n$

③ $$a_i^T \tilde{x} = a_i^T x + \delta a_i^T v$$
$$\geq a_i^T x - \min_{j \notin [n]} (a_j^T x - b_j)$$
$$\geq b_i$$

for $i \notin [n]$

- $$c^T \tilde{x} = c^T x + \delta c^T v$$
$$= p^* + \delta \sum_{i=1}^{n} y_i a_i^T v$$
$$= p^* + \delta \underbrace{y_1}_{<0} \underbrace{a_1^T v}_{>0}$$
$$< p^*$$

CONTRADICTION

$\blacksquare$

Strong Duality Thm

$$\min c^T x \quad = \quad \max b^T y$$
$$Ax \geq b \qquad\qquad y^T A \leq c$$
$$x \geq 0 \qquad\qquad y \geq 0$$

- Let (x, y) be any pair of optimum solutions.

- $0 = c^T x - y^T b$

$$\geq (y^T A) x - y^T A x = 0$$

$\therefore$ All inequalities are strict eq.

$\therefore$ (1) $c^T x = y^T A x$

$$\Rightarrow \quad x \cdot (y^T A - c) = 0$$

$$\Rightarrow \forall i = 1 .. n, \begin{cases} x_i = 0 & \text{or} \\ (y^T A)_i = c_i \end{cases}$$

(2) $y^T b = y^T A x$

$$\Rightarrow y^T(b - Ax) = 0$$

$$\Rightarrow \forall i = 1..m: \begin{cases} y_i = 0 & , \text{or} \\ b_i = (Ax)_i \end{cases}$$

Contrapositively,

If $y_i > 0$, then $(Ax)_i = b_i$

Some dual value +ve $\Rightarrow$ corr Constr. is tight.

If $x_i > 0$, then $(y^T A)_i = c_i$

The primal-dual method

Vertex-Cover:

$$\min \sum_v c_v x_v \qquad\qquad \max \sum_e y_e$$

$$\forall (u,v): x_u + x_v \geq 1 \qquad\qquad \forall u: y(\delta(u)) \leq c_u$$

$$\forall u \in V \quad x_u \geq 0 \qquad\qquad y \geq 0$$

$$\min \sum_v c_v x_v \qquad \Big| \qquad \max \sum_e y_e$$

$$\forall (u,v): \; x_u + x_v \geq 1 \qquad \Big| \qquad \forall u: \; y(\delta(u)) \leq c_u$$

$$\forall u \in V \quad x_u \geq 0 \qquad \Big| \qquad y \geq 0$$

Methodology:-

- Start with a feasible dual soln.

  Note : val (dual) $\leq$ LP $\leq$ OPT

- Try to raise dual variables. so that val (dual) $\uparrow$

- Some dual constraint will go tight. Take that as a cue to "pick" the corr. primal var.

- Challenge:- Pay for primal cost increase with dual increase.

Algorithm for VC

finally a VC

① Start with $y_e = 0 \;\; \forall e$, $I = \emptyset$

(2) Initially all edges are active A.

(3) Raise duals y_e $\forall e \in E$ @ the "same rate" $\frac{dy_e}{dt} = 1$, say

till some vertex v becomes tight.

(4) "Freeze" duals of all edges inc. on v. Remove them from active set

(5) Add v to I

(6) Go back to step 3 till all edges are covered.

Analysis - Let (y, I) be

— By definition of algo, I is a VC. and y is feas. dual.

$- \quad \forall v \in I: \quad c(v) = \sum y_e$

$e \sim v$

- $\therefore$ $$\mathrm{ALG} = \sum_{v} c(v) = \sum_{v \in I} \sum_{e \sim v} y_e$$

$$= \sum_{e} y_e \, |e \cap I|$$

$$\leq 2 \sum_{e} y_e \leq 2\mathrm{LP} \qquad \blacksquare$$