CS 49/149: Approximation Algorithms

Problem set 6. Due: 26th May, 6:59pm

General small print: Please submit all homework electronically in PDF format ideally typeset using LaTeX. You need to submit only the problems above the line. Please try to be concise – as a rule of thumb do not take more than 1 (LaTeX-ed) page for a solution. We highly encourage students to also do the problems below the line for a better understanding of the course material.

Topics in this HW: The Dual LP, Randomized Rounding

Problem 1. Write duals of the following LPs.

(a.) **(3 points)** We are given n jobs and m machines and each machine has a capacity 1. The running time of job j on machine i is p_{ij} and the profit of assigning job j to machine i is v_{ij}. The following is an LP relaxation of the problem: find an assignment of jobs to machines such that each machine gets total load at most 1, and the total profit is maximized. Below i index machines and j index jobs.

$$\max \sum_{i,j} v_{ij}x_{ij} : \quad \forall i, \sum_j p_{ij}x_{ij} \leq 1, \quad \forall j, \sum_i x_{ij} \leq 1, \;\; x \geq 0$$

(b.) **(4 points)** The min-cut LP we looked in class. The primal LP has distance variables $d(u,v)$ for every pair of vertices.

$$\min \sum_{(u,v)\in E} c(u,v)d(u,v) : \;\; \forall (u,v,w) : d(u,w) \leq d(u,v) + d(v,w), \quad d(s,t) = 1, \quad d \geq 0$$

Is there a relation between this dual and the max-flow LP with $c(u,v)$'s as capacities?

Problem 2. Recall the spanning tree polytope we did in class.

$$\{x \in \mathbb{R}^E : \sum_{e\in E} x_e = |V| - 1; \quad \forall S \subseteq V, \; x(E[S]) \leq |S| - 1, \quad \forall e \in E, \; 0 \leq x_e \leq 1\}$$

(a.) Consider the LP which minimizes $\sum_e c(e)x_e$ over the above polytope. Write the dual of this LP. **(1 point)**

(b.) Write the complementary slackness conditions.**(2 points)**

(c.) Consider the run of Kruskal's algorithm to find the minimum spanning tree: if $\{e_1, \ldots, e_m\}$ were the edges in non-decreasing order, then the algorithm picks edges in this order discarding any edge which forms a cycle. Construct a feasible dual solution whose cost is the same as that of what Kruskal's algorithm returns. Hint: Since you know that the Kruskal's solution is indeed the optimum MST, complementary slackness conditions should point you towards which sets should have non-zero dual values in the optimum dual solution.

Problem 3. Recall the max-k-coverage problem: given m sets $S_1, \ldots, S_m$ with each $S_i \subseteq \{1, \ldots, n\}$, pick k sets such that the size of their union is maximized. In this problem we wish to solve this via randomized rounding.

(a.) **(2 points)** Write an LP-relaxation for the problem with variables x_i to indicate set S_i is picked or not. You may also need variables z_j for $j \in [n]$ to indicate whether element j has been covered or not.

(b.) **(5 points)** Design and analyze a randomized approximation algorithm for the problem which attains a factor $(1 - 1/e)$.

You may find the following useful: the function $g(t) := 1 - e^{-t}$ is concave, and so for any $\theta \in (0, 1)$, we have $g(\theta) \geq \theta g(1) + (1 - \theta) g(0) = (1 - 1/e) \cdot \theta$.