CS 30: Discrete Math in CS (Winter 2019): Lecture 13

Date: 25th January, 2019 (Friday)
Topic: Predicate Logic & Quantifiers
Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.
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1. **Predicates.** A *predicate* is a function $P(x)$ with a variable x (or multiple variables such as x, y) where (a) the variable takes a value from a certain set (say, integers for now), and (b) given a fixed value for the variable x, the predicate $P(x)$ becomes a proposition.

   Here is an example.
   $$P(x) = (x \text{ is an odd number})$$
   where the variable x takes a value among the integers. Given a fixed value for x, say $x = 3$, we get a proposition $P(3)$ which is true. On the other hand, $P(2)$ is false.

   As mentioned above, predicates can take multiple variables. For instance,
   $$P(x, y) = (x + y \text{ is an even number})$$
   is a predicate where both its variables take a value among the integers. $P(2, 3)$ is false, but $P(1, 9)$ is true. ✍

   Exercise: *Define a predicate of your own.*

2. **Quantifiers.** A predicate by itself is neither true nor false. The power of predicates arise when used with *quantifiers*. There are two quantifiers:

   $\forall$, which stands for "for all" and $\exists$, which stands for "there exists"

   Using this we can get new propositions which take true or false values. Formally,

   Remark: *Given any predicate $P(\cdot)$ and a set S such that $P(x)$ takes value* true *or* false *for any $x \in S$, the following are propositions in* predicate logic*:*
   $$\forall x \in S : \; P(x) \qquad \exists x \in S : \; P(x)$$
   The set S is called the domain of discourse*.*

   For example,
   $$\phi := \forall x \in \mathbb{Z} : P(x) \quad \text{where, } P(x) = (x \text{ is an odd number})$$
   is a statement which takes a value true or false. The set of integers $\mathbb{Z}$ is the *domain of discourse*. It is true if *for every* fixed $x \in \mathbb{Z}$, that is, every fixed integer x, the proposition $P(x)$ is true. As you can see, ϕ takes the value false (because not every integer is odd.)

   The following is also a proposition.
   $$\psi := \exists x \in \mathbb{Z} : P(x) \quad \text{where, } P(x) = (x \text{ is an odd number})$$
   It takes the value true if *for some* fixed $x \in \mathbb{Z}$, the proposition $P(x)$ is true. As you can see, ψ takes the value true (since some integer, say 3, is odd.)

Thinking of Quantifiers as for-loops. It may be useful to think of the formula defined by quantifiers and predicates as being evaluations of for loops.

$\forall x \in S : P(x) :=$

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for x ∈ S do:
    ▷ Iterating over the set S, In Python, think of S as a list.
    if P(x) is false then:
        return false
return true. ▷ If not returned false yet.
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$\exists x \in S : P(x) :=$

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for x ∈ S do:
    ▷ Iterating over the set S, In Python, think of S as a list.
    if P(x) is true then:
        return true
return false. ▷ If not returned true yet.
```

3. **Thinking of Quantifiers as ANDs and ORs.** Suppose $S = \{x_1, x_2, \ldots, x_k\}$ is a finite domain of discourse. Then, another way of thinking about quantifiers is

$$\forall x \in S : P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_k) \qquad \exists x \in S : P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_k)$$

Note that when the set S is infinite (such as natural numbers or integers), then the above is just a tool to think conceptually. Be wary in using this in proofs.

4. **Examples of statements expressed in Predicate Logic.** Below we show the expressiveness of predicate logic (quantifiers + predicates) using some examples of statements and they expressed in predicate logic. This helps formalize English statements.

   - *A number is divisible by* 4 *if and only if its last two digits are.*

     We define two predicates $P(n) = (n \text{ is divisible by } 4)$ and $Q(n) = (\text{Last two digits of } n \text{ are divisible by } 4)$. The variables for both predicates takes value in natural numbers. The above statement expressed in predicate logic is

     $$\forall n \in \mathbb{N} : ((\neg P(n) \vee Q(n)) \wedge (\neg Q(n) \vee P(n)))$$

   - *An irrational number raised to power an irrational number can be a rational number.*

     We define a predicate $P(z) = (z \text{ is an irrational number})$, which takes the variable in *real numbers*. The above statement can be written in predicate logic as

     $$\exists x \in \mathbb{R}, y \in \mathbb{R} : (P(x) \wedge P(y) \wedge \neg P(x^y))$$

Exercise: *Write the following in predicate logic. Clearly define your predicates.*

- *All prime numbers at least* 3 *are odd.*
- *Any perfect square is either odd or it has a* $0, 4,$ *or* 6 *in the units place.*

5. **Negations of statements in predicate logic.** Say you would like to prove a statement such as one of the examples given above. Suppose you decide to prove the statement by contradiction. The first step is to understand what the contradiction even means, that is, you need to figure out the negation of the statement you want to prove.

$$\neg(\forall x \in S : P(x)) \equiv \exists x \in S : \neg P(x) \qquad \text{(Negation of a } \forall\text{)}$$

$$\neg(\exists x \in S : P(x)) \equiv \forall x \in S : \neg P(x) \qquad \text{(Negation of a } \exists\text{)}$$

One way to see why, say (Negation of a ∀), is true, is to use the "propositional logic view" of predicate logic statements, and apply De Morgan's Law. More formally, suppose $S = \{x_1, \ldots, x_k\}$ is finite. Then,

$$\forall x \in S : P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_k)$$

and therefore,

$$\neg\forall x \in S : P(x) \equiv \quad \neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_k) \quad \equiv \exists x \in S : \neg P(x)$$

A (correct) proof of (Negation of a ∀) which doesn't deal with infinities is the following. We show whenever the formula $\neg(\forall x \in S : P(x))$ takes the value true, the formula $\exists x \in S : \neg P(x)$ takes the value true, and vice-versa.

The formula $\neg(\forall x \in S : P(x))$ takes value true implies the formula $\forall x \in S : P(x)$ takes the value false. If the formula $\forall x \in S : P(x)$ takes value false, there must exist an $a \in S$ such that $P(a)$ is false which implies $\neg P(a)$ is true, that is $\exists x \in S : \neg P(x)$ is true. The vice-versa direction, which is crucial, is left as an exercise to the reader. ✍

Exercise: *Do the exercise.*

✍

Exercise: *Prove* (Negation of a ∃)

This allows us to take say the negation of the statement "An irrational number raised to power an irrational number can be a rational number." In predicate logic, the negation is

$$\neg(\exists x \in \mathbb{R}, y \in \mathbb{R} : (P(x) \wedge P(y) \wedge \neg P(x^y))) \equiv \forall x \in \mathbb{R}, y \in \mathbb{R} : (\neg P(x) \vee \neg P(y) \vee P(x^y))$$

which we can read out in English as "For any two real numbers x, y, either x is rational or y is rational or x^y is irrational."

This may or may not be the converse of the statement given that you would've thought of (For me it was not). But let's rewrite it slightly to get

$$\begin{aligned}\forall x \in \mathbb{R}, y \in \mathbb{R} : (\neg P(x) \vee \neg P(y) \vee P(x^y)) &\equiv \forall x \in \mathbb{R}, y \in \mathbb{R} : (\neg(P(x) \wedge P(y)) \vee P(x^y)) \\ &\equiv \forall x \in \mathbb{R}, y \in \mathbb{R} : ((P(x) \wedge P(y)) \Rightarrow \neg P(x^y))\end{aligned}$$

where we have used that $p \Rightarrow q$ is equivalent to $\neg p \vee q$. Now, the English version of this statement is "For any two real numbers x, y, if both x and y are irrational, then x^y is irrational." This was the converse I would've thought of.

6. **Nested Quantifiers.** In the example above regarding irrational numbers, we had two variables x, y and we used $\forall$ implicitly for both. However, there are some instances where we have $\forall$ for one variable and $\exists$ for the other. For example, consider the statement "For every integer x, there is an integer y which is bigger than it."

   In predicate logic we will write this as

   $$\phi := \forall x \in \mathbb{Z} \ \exists y \in \mathbb{Z} : \ P(x, y)$$

   where $P(x, y)$ is the predicate taking the value true if $y > x$. What is the truth value of this formula? It is true. A proof of this is like a game between you and an adversary. The adversary is the $\forall$ person who gives x. Once x is fixed, our job is to find a $y \in \mathbb{Z}$ such that $P(x, y)$ is true. $y = x + 1$ suffices.

   Note the **order** of quantifiers. If we flip this order, we get something completely different.

   $$\psi := \exists y \in \mathbb{Z} \ \forall x \in \mathbb{Z} : \ P(x, y)$$

   which in English translates to "There exists an integer y which is bigger than every integer x." The value of this statement is false. Once again, you can play the game with the adversary who claims this formula is true. To do so, she produces the integer y^* and claims $\forall x \in \mathbb{Z}$: $P(x, y^*)$ is true. But then you show $x = y^* + 1$ for which $P(x, y)$ is false.

7. **Negations of Nested Quantifiers.** The negation of the formula

   $$\phi := \forall x \in \mathbb{Z} \ \exists y \in \mathbb{Z} : \ P(x, y)$$

   can be written using the above rules as

   $$\neg\phi \equiv \qquad \exists x \in \mathbb{Z} \neg \left(\exists y \in \mathbb{Z} : \ P(x, y)\right) \qquad \equiv \exists x \in \mathbb{Z} \ \forall y \in \mathbb{Z} : \neg P(x, y)$$

8. **More examples of statements expressed in predicate logic.**

   - *Every non-zero real number has a reciprocal.*
     We need to show that for any real number x such that $x \neq 0$, there is some other real number y such that $xy = 1$. To do so, we define the predicate $P(x, y) = \mathsf{true}$, if $xy = 1 \quad = \mathsf{false}$, otherwise. The above statement is then expressed as

     $$x = 0 \ \vee \ \Big(\forall x \in \mathbb{R} \, \exists y \in \mathbb{R} : P(x, y) \Big)$$

   - *There are infinitely many primes.*
     This is slightly interesting. What does it mean to have infinitely many primes? It means that for any number N, there is some prime bigger than N. This we can express using the predicate $P(n)$ which is true if n is prime and false otherwise. The above statement is then expressed as

     $$\forall N \in \mathbb{N} \ \exists n \in \mathbb{N} : \ \Big(n > N \wedge P(n)\Big)$$

We could do this another way. We could interpret the infinitude of prime numbers as: for any prime number, there is another prime number larger than it. The above would then be expressed as:

$$\forall N \in \mathbb{N} : \quad \Big(P(N) \Rightarrow \Big(\exists n \in \mathbb{N} : \Big(n > N \wedge P(n)\Big)\Big)\Big)$$

Note that $P(N)$ is checking whether N is a prime or not. If N is not a prime, then $P(N)$ would be false, and thus the implication would be true. So, in essence, the above statement is "for-all"-ing only over the prime numbers. Note also how the second quantifier is "deep" within the first. ✍

Exercise: *Is the above statement equivalent to*

$$\forall N \in \mathbb{N}\, \exists n \in \mathbb{N} : \quad (P(N) \Rightarrow ((n > N) \wedge P(n)))$$

Use the conversion of $\forall, \exists$ *into ANDs and ORs to check. If not equivalent, say why.*