CS 30: Discrete Math in CS (Winter 2019): Lecture 28

Date: 1st March, 2019 (Friday)
Topic: Graphs: Bipartite Graphs, Matchings
Disclaimer: These notes have not gone through scrutiny and in all probability contain errors. Please discuss in Piazza/email errors to deeparnab@dartmouth.edu

1. **Bipartite Graphs and Matchings.**

A graph $G = (V, E)$ is ***bipartite*** if there is partition of $V = L \cup R$ such that $L \cap R = \emptyset$ and for every edge $e = (u, v) \in R$, we have $|\{u, v\} \cap L| = |\{u, v\} \cap R| = 1$. That is, every edge has exactly one endpoint in L and exactly one endpoint in R.

Bipartite graphs are very useful objects to denote relations between two classes of objects: agents-items, girls-boys, students-courses, etc. In the UGP (over three problems) you will prove the following:

Theorem 1. A graph $G = (V, E)$ is bipartite if and only if it has no odd-length cycles.

A ***matching*** M in a graph is a subset of edges $M \subseteq E$ such that for any $e, e' \in M$, $e \cap e' = \emptyset$. That is, M is a collection of edges which do not share end points. A vertex $v \in V$ participates in the matching M if there is an edge in M which is incident to v.

These are *fundamental objects* and have numerous applications. For instance, in economics, where the bipartite graph contains agents on one side and items on the other, where the edges represent desirable items, and each agent has only a demand of one item, then a matching corresponds to an allocation of desirable items to these agents.

A matching is a ***perfect matching*** if every vertex of V appears in some edge of the matching.

In this section, we look at matchings in bipartite graphs. To this end, fix a bipartite graph $G = (V, E)$ where V has been partitioned in to $L \cup R$. We say that a matching $M \subseteq E$ is an L-matching if all vertices in L participate in M. Similarly, a matching M is an R-matching if all vertices in R participate in M. The following amazing theorem gives a *necessary* and *sufficient* condition of when an L-matching exists in a bipartite graph $G = (L \cup R, E)$.

To describe this, recall the notion of a ***neighborhood***: $N_G(v) := \{u \in V : (u, v) \in E\}$. Indeed, we can *generalize* this definition to *subsets of vertices*. Given any subset $S \subseteq V$, we define

$$N_G(S) := \{v \in V : \ \exists w \in S \text{ such that} (v, w) \in E\}$$

In English, $N_G(S)$ is the set of vertices which have at least one neighboring vertex in S. Another way of looking at it is $N_G(S) = \cup_{v \in S} N_G(v)$.

Theorem 2 (Hall's Theorem)**.** Let $G = (V, E)$ be a bipartite graph with $V = L \cup R$. Then, G has an L-matching if and only if

$$\text{For every subset } S \subseteq L, \ |N_G(S)| \geq |S|$$

Corollary 1. A bipartite graph $G = (L \cup R, E)$ has a *perfect matching* if and only for any $S \subseteq L$ or $S \subseteq R$ we have $|N_G(S)| \geq |S|$.

Applications.

- *Left-dominant bipartite graphs.* A bipartite graph $G = (L \cup R, E)$ is *left dominant* if $\deg_G(x) \geq \deg_G(y)$ for any $x \in L$ and any $y \in R$. The Hall's theorem shows that any left-dominant graph has an L-matching.

  To see this, it suffices to show that for any subset $S \subseteq L$, $|N_G(S)| \geq |S|$. Let $d_L := \min_{v \in L} \deg_G(v)$ and $d_R := \max_{u \in R} \deg_G(u)$. G being left-dominant means $d_L \geq d_R$. Now consider the graph H induced by $(S \cup N_G(S))$; let $H = (S \cup N_G(S), E_S)$. Note that $\deg_H(u) = \deg_G(u)$ for all $u \in S$, and $\deg_H(v) \leq \deg_G(v)$ for all $v \in N_G(S)$.

  Next note,

  (a) $|E_S| = \sum_{u \in S} \deg_H(v) = \sum_{u \in S} \deg_G(u) \geq d_L \cdot |S|$, and

  (b) $|E_S| = \sum_{w \in N_G(S)} \deg_H(w) \leq \sum_{w \in N_G(S)} \deg_G(w) \leq d_R \cdot |N_G(S)|$.

  Thus, we get,
  $$d_L \cdot |S| \leq |E_S| \leq d_R \cdot |N_G(S)|$$
  implying $|N_G(S)| \geq |S|$.

  Exercise: *Prove that a* regular *bipartite graph always has a perfect matching.*

- *Completeing Latin Rectangles to Latin Squares.*

  A *Latin rectangle* is an $r \times n$ matrix with $r \leq n$. Each entry of the matrix has numbers from $\{1, 2, \ldots, n\}$. The constraint is that any row and any column has *no repeating entry*.

  So, for example, the following are examples of Latin rectangles; one is a 2×5 and the other is a 3×5.

1	2	3	4	5
2	3	4	5	1

1	2	3	4	5
3	1	4	5	2
2	5	1	4	3

  An $n \times n$ Latin rectangle is called a Latin square. A *completion* of an $r \times n$ Latin rectangle is an $n \times n$ Latin square whose first r rows is the Latin rectangle. Can every Latin rectangle be completed?

  Theorem 3. Every Latin rectangle can be completed.

  Proof. Let us fix an $r \times n$ Latin rectangle T. We show how to construct an $(r+1) \times n$ Latin rectangle whose first r rows are the rows of T. We can then repeat this till we get our desired Latin square.

  We do so by using Hall's theorem!

  First, we construct a bipartite graph $G = (L \cup R, E)$. Both L and R is the set $\{1, 2, \ldots, n\}$. The vertex $i \in L$ corresponds to the entry of the ith column of the $(r+1)$th row. The vertex $j \in R$ corresponds to the number j. We have an edge (i, j) in E if the number i *does not* appear in the ith column of T. That is, the number i is a feasible candidate to be put in the ith column of the $(r + 1)$th row. This completes the description of the graph.

Now observe: if G has a L-saturated matching, then we can fill the $(r+1)$th row. Indeed, if the matching has the edge (i, j) we put the number j in the ith column of the $(r+1)$th row.

To show that G has a perfect matching, we show that G is a left-dominant graph. Fix a vertex $x \in L$. What is $\deg_G(x)$? For each column of the $(r + 1)$th row, exactly r numbers are disallowed and so $(n - r)$ numbers are allowed. Thus, $\deg_G(x) = n - r$.

Now fix a vertex $y \in R$. What is $\deg_G(y)$? This is the number of columns of the $(r+1)$th row in which the number y can be put. This is precisely the columns in which y *doesn't* appear. But y appears in r different columns, and thus the number of columns free for y is also $(n - r)$. Thus, not only is G left-dominant, but rather it is a ***regular*** graph; all degrees are equal.

Therefore, G has a L-dominant matching. And thus, a Latin rectangle can be completed to a Latin square. $\square$