CS 30: Discrete Math in CS (Winter 2019): Lecture 3

Date: 7th January, 2019 (Monday)
Topic: Sets and Functions
Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.
Please discuss in Piazza/email errors to deeparnab@dartmouth.edu

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1 Sets

- **Definition.** A set is an *unordered* collection of objects. These objects are called *elements* of the set. These elements could be *anything*, for instance, the element of a set could be a number, could be a string, could be tuples of numbers, and in fact can be other sets!

- $\in$ **notation.** An element x of S is said to satisfy $x \in S$. If x is not an element of S, we denote it as $x \notin S$.

- **How to describe a set?** A set is described either by explicitly writing down the elements, such as
$$S = \{1, 3, 5, 7, 9\} \quad \text{or} \quad T = \{\text{apple, banana, volcano}, 100\}$$
This is called the *roster notation*.

  Or, a set is described implicitly by stating some rule which the elements follow, such as
$$S = \{n : n \text{ is an odd number less than } 10\} \quad \text{or} \quad W = \{x^2 : x \text{ is an integer and } 1 \leq x \leq 5\}$$
This is called the *set-builder notation*.

  The sets S described in the above two examples correspond to the same set. The set W, written explicitly in the roster notation, looks like $W = \{1, 4, 9, 16, 25\}$.

  > **Remark:** *Caution: Unless otherwise explicitly mentioned, duplicate items are removed from a set. For example, consider the set $A = \{x^2 : -2 \leq x \leq 2\}$ in the set-builder notation. In the roster notation, this set is $\{0, 1, 4\}$ and* **not** *$\{4, 1, 0, 1, 4\}$. Sometimes one may allow duplicates, but in that case the set will be* explicitly *called a* multiset.

- **Cardinality of a set.** The *cardinality* of a set S is denoted as $|S|$ is the number of elements in the set. For example if $A = \{\text{apple, banana, avocado}\}$, then $|A| = 3$. ✍

  > **Exercise:** *What is $|A|$ when $A = \{x^2 : -3 \leq x \leq 3\}$?*

  If the set S has only finitely many elements, then $|S|$ is a finite number, and S is called a *finite* set.

  $|S|$ could be ∞ in which case the set is called an infinite set.

- **Famous examples of Infinite Sets.** $\mathbb{N}$, the set of all natural numbers; $\mathbb{Z}$, the set of all integers; $\mathbb{Q}$, the set of all rational numbers, $\mathbb{R}$, the set of all real numbers; and $\mathcal{P}$, the set of all computer programs written in Python.

- **Empty Set.** There is only one set which contains no elements and that set is called the *empty set*. It is denoted as $\emptyset$ or $\{\}$.

- **Subsets and Supersets.** A *subset* P of a set S is another set such that every element of P is an element of S. In that case, the notation used is $P \subset S$. In case P is a subset and not equal to S, it is called a *proper subset*. It is denoted as $P \subsetneq S$.

  For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, then $B \subset A$.

  Remark: *For any set A, the set A itself is also a subset of A. That is, $A \subset A$. It is not a proper subset.*

  Remark: *The empty set $\emptyset$ is a subset of* any set.

  ✍

  Exercise: *Write down* all *subsets of the sets $S = \{1, 2\}$, $T = \{1, 2, 3\}$ and $U = \{1, 2, 3, 4\}$. Do you see a pattern in the number of subsets?*

  If $A \subset B$, then B is called a *superset* of A. This is denoted as $B \supset A$.

- **Set Operations.**

  - **Union.** Given two sets A and B, the *union* $A \cup B$ is the set containing all elements which are either in A, or in B, or both. For example, if

    $$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \quad \text{then } A \cup B = \{1, 2, 3, 4, 7, 9, 10\}$$

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    Exercise: *Can there be sets A and B such that $|A \cup B| > |A| + |B|$?*

  - **Intersection.** Given two sets A and B, the *intersection* $A \cap B$ is the set containing all elements which are in *both* in A and in B. For example, if

    $$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \quad \text{then } A \cap B = \{4, 7, 10\}$$

    Two sets A and B are called *disjoint* if $A \cap B = \emptyset$.

    Theorem 1. If A and B are two disjoint finite sets, then $|A \cup B| = |A| + |B|$.

    Proof. We will give a computer-sciency proof of the above theorem. Consider maintaining three *counters* $C_{A \cup B}$, C_A and C_B. Initially all counters are set to 0. Next, we run the following code. We consider the elements of $A \cup B$ in a list (say), and then we iterate over the elements e in this list. For each element e we do the following:

    1. We increment $C_{A \cup B}$ by 1.
    2. If $e \in A$: we increment C_A by 1.
    3. If $e \in B$: we increment C_B by 1.

Step 1 implies that at the end of the for-loop, $C_{A \cup B}$ will be set to $|A \cup B|$. Since A and B are disjoint, no element e is in both A and B. Furthermore, every element $e \in A \cup B$ has to be in either A or B. Therefore, in every iteration of the for-loop *exactly* one of C_A or C_B is incremented. Therefore, at the end of the for-loop, $C_A + C_B = C_{A \cup B}$.

Finally, we assert that $C_A = |A|$ and $C_B = |B|$. To see the former, note that

1. We never increment C_A unless we see an element of A,
2. We never see the same element e of A twice since $A \cup B$ has distinct elements, and
3. Every element of A is seen in the for-loop since every element of A is also in $A \cup B$.

(1) and (2) imply $C_A \leq |A|$, and (3) implies $C_A \geq |A|$. Thus, we get $C_A = |A|$.

We can apply *exactly the same argument for C_B and B* (you should try it without looking at the notes above) to get $C_B = |B|$. The proof now follows since

$$|A \cup B| = C_{A \cup B} = C_A + C_B = |A| + |B|.$$

□

Remark: *Please note the subtleties involved in proving $C_A = |A|$. To appreciate this, consider looping over a* multiset with duplicates *or over some other subset C.*

– **Difference.** Given two sets A and B, the *set difference* $A \setminus B$ are all the elements in A which are *not in* B and $B \setminus A$ are the elements in B which are not in A. For example, if

$$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \quad \text{then } A \setminus B = \{1, 3\} \quad \text{and} \quad B \setminus A = \{2, 9\}$$

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Exercise: *Is $A \setminus B = B \setminus A$? Can they ever be equal?*

Remark: *A couple of useful observations:*

1. *A and $B \setminus A$ are* disjoint *since $B \setminus A$ doesn't contain elements of A.*
2. *In particular, this implies $(A \cap B)$ and $B \setminus A$ are disjoint since $A \cap B \subseteq A$.*
3. *$A \cup (B \setminus A) = A \cup B$. This is because every element of $A \cup B$ is either in A, and if not in A, must be in $B \setminus A$.*
4. *$(A \cap B) \cup (B \setminus A) = B$. This is because every element of B is either in A (in which case it is in $A \cap B$) or in $B \setminus A$.*

Theorem 2 (Inclusion-Exclusion (baby version))**.** For any two finite sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. Since $A \cup B = A \cup (B \setminus A)$ and since A and $B \setminus A$ are disjoint, from Theorem 1 we get

$$|A \cup B| = |A| + |B \setminus A| \tag{1}$$

Since $B = (A \cap B) \cup (B \setminus A)$ and since $(A \cap B)$ and $B \setminus A)$ are disjoint, from Theorem 1 we get

$$|B| = |A \cap B| + |B \setminus A| \tag{2}$$

Subtracting (2) from (1), we get

$$|A \cup B| - |B| = |A| - |A \cap B|$$

The theorem follows by taking $|B|$ to the other side. □

– **Cartesian Product** Given two sets A and B, the *Cartesian product* $A \times B$ is another set C whose elements are *tuples* of the form (a, b) where $a \in A$ and $b \in B$. That is, $A \times B := \{(a, b) : a \in A \textit{ and } b \in B\}$. For example, if

$$A = \{1, 3\} \text{ and } B = \{2, 4, 7\}, \quad \text{then} A \times B = \{(1, 2), (1, 4), (1, 7), (3, 2), (3, 4), (3, 7)\}$$

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Exercise: *If $|A| = a$ and $|B| = b$, then what is $|A \times B|$?*