CS 30: Discrete Math in CS (Winter 2019): Lecture 4

Date: 9th January, 2019 (Wednesday)
Topic: Functions and Countable Sets

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors. Please discuss in Piazza/email errors to deeparnab@dartmouth.edu

1 Functions

- **Definition.** A function is a *mapping* from one set to another. The first set is called the *domain* of the function, and the second set is called the *co-domain*. For *every* element in the domain, a function assigns a unique element in the co-domain.

  Notationally, this is represented as
  $$f : A \to B$$
  where A is the set indicating the domain, and B is the set indicating the co-domain. For every $a \in A$, the function maps the value of $a \mapsto f(a)$ where $f(a) \in B$.

  For example, suppose
  $$A = \{1, 2, 3\}, \text{ and } B = \{5, 6\}, \text{ then the map } f(1) = 5, f(2) = 5, f(3) = 6 \text{ is a valid function.}$$

  The *range* of the function is the subset of the co-domain which are *actually mapped to*. That is, $b \in B$ is in the range if and only if there is some element $a \in A$ such that $f(a) = b$.

  More Examples.

  - $f(x) = x^2$ is a function whose domain is $\mathbb{R}$, the set of read numbers, and so is the co-domain. The range, however, is the set of non-negative real numbers (sometimes denoted as $\mathbb{R}_+$).
  - $f(x) = \sin x$ is a function whose domain is $\mathbb{R}$ and the range is the interval $[-1, 1]$.
  - A (deterministic) computer program/algorithm is also a function; its domain is the set of possible inputs and its range is the set of possible outputs. ✍

  Exercise: *Given a set $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, describe a function f whose range is $\{5\}$, and describe a function g whose range is $\{4, 6\}$. Just to get a feel, how many functions can you describe of the first form (whose range is $\{5\}$), and how many functions can you describe of the second form?*

- **Sur-, In-, Bi- jective functions.** A function $f : A \to B$ is

  - *surjective*, if the range is the same as the co-domain. That is, for every element $b \in B$ there exists some $a \in A$ such that $f(a) = b$. Such functions are also called *onto* functions. For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, and consider the function $f : A \to B$ with $f(1) = 5$, $f(2) = 5$, and $f(3) = 6$. Then, f is surjective. This is because for $5 \in B$ there is $1 \in A$ such that $f(1) = 5$ and for $6 \in B$ there is a $3 \in A$ such that $f(3) = 6$. ✍

> **Exercise:** *If A and B are* finite sets*, and $f : A \to B$ is a surjective function, can you show $|B| \leq |A|$?*

- *injective*, if there are no collisions. That is, for any two elements $a \neq a' \in A$, we have $f(a) \neq f(a')$. Such functions are also called *one-to-one* functions. For an injective function, one can define $f^{-1}(b)$ for all b in the range of f.

  For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$, and consider the function $f : A \to B$ with $f(1) = 5, f(2) = 6$, and $f(3) = 8$. Then, f is injective. This is because $f(1), f(2), f(3)$ are all distinct numbers.

  > **Exercise:** *If A and B are* finite sets*, and $f : A \to B$ is an injective function, can you show $|A| \leq |B|$?*

  Injective functions have *inverses*. Formally, given any injective function $f : A \to B$, we can define a function $f^{-1} : B \to A \cup \{\perp\}$ as follows

$$f(b) = \begin{cases} a & \text{if } a \text{ is the } \textit{unique } a \in A \text{ with } f(a) = b. \\ \perp & \text{otherwise, that is } f(a) \neq b \text{ for all } a \in A. \end{cases}$$

  Sometimes people define a different inverse function where instead of adding the $\perp$ to the co-domain, they only consider the *range* of f as the domain. That is, the following is a valid function $g : \mathsf{range(f)} \to A$ where g maps $b \in \mathsf{range(f)} \mapsto a$ where a is the unique element in A with $f(a) = b$. In particular, when f is bijective (to be defined below), this is the definition of the inverse and the $\perp$ is not used.

- *bijective*, if the function is both surjective and injective.

  For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the function $f(x) = 2x$ defined over the domain A and co-domain B is a bijective function. Can you see why?

  > **Exercise:** *If A and B are* finite sets*, and $f : A \to B$ is a bijective function, can you show $|B| = |A|$?*

2 Countable Sets

- **Definition.** A set S is *countable* if there exists an injective function $f : S \to \mathbb{N}$.

  It is called so because the elements of S can be ordered and counted one at a time (although the counting may never finish). More precisely, using f we can devise an ordering of the elements an S and an algorithm which for any natural number k gives the kth number in the ordering. Note that since f is injective, for every $n \in \mathbb{N}$, either $f^{-1}(n)$ doesn't exist, or $f^{-1}(n)$ is a well-defined element in S. The following code prints this sequence.

  for $n = 1, 2, 3, \ldots$ **do**: ▷ $n \in \mathbb{N}$
      if There exists some $a \in S$ such that $f(a) = n$ **then**: ▷ i.e. $f^{-1}(n) \in S$
          Print a

- **Examples**
  - *Finite sets* are almost trivially countable. If a set S is finite and $|S| = k$, then the elements of S can be renamed as $\{e_1, e_2, \ldots, e_k\}$. The injective function $f(e_i) = i$ implies S is countable.

    Infinite sets can also be countable. $\mathbb{N}$ is countable by definition. But there are many more interesting examples.
  - *Set of Integers.* The set $\mathbb{Z}$ is countable. To see this, consider the following function $f : \mathbb{Z} \to \mathbb{N}$. If $x > 0$, then $f(x) = 2x$. If $x \leq 0$, then $f(x) = 2(-x) + 1$. Note that the co-domain of this function is indeed the natural numbers.

    For instance, $f(2) = 4$, $f(-2) = 5$, and $f(0) = 1$.

    Claim 1. The function $f : \mathbb{Z} \to \mathbb{N}$ defined above is injective.

    Proof. To see this is injective, we need to show $f(x) \neq f(y)$ for two integers $x \neq y$. We may assume, without loss of generality, $x < y$. If both x and y are positive, then $f(x) = 2x < 2y = f(y)$. Similarly, if both are non-negative, then we get $f(x) = -2x + 1 > -2y + 1 = f(y)$. The only other case is x is non-negative and y is positive. In this case, $f(x)$ is odd while $f(y)$ is even. $\square$

    If we use the above algorithm to figure out the ordering of $\mathbb{Z}$, we get:

    $$(0, 1, -1, 2, -2, 3, -3, 4, -4, \cdots)$$