Lecture 13

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Randomized Approximation Algorithms

We will look at algorithms which are allowed to toss coins as they go. Such algorithms will have an error probability – a small chance with which the answer will be WRONG.

It's important that the ANALYSIS of such algorithms can BOUND the FAILURE probability.

Ideal Definition

A randomized approximation algorithm $\mathcal{A}$ for a (minimization) problem Π takes as input instance $I \in \Pi$ and an error parameter $\delta > 0$. It is an α-appx factor algo if

$\forall \delta > 0, \forall I \in \Pi$

$$\Pr[\text{cost}(A(I)) \leq \alpha \cdot OPT(I)] \geq 1 - \delta$$

and $A(I)$ runs in time $\text{poly}(n) \cdot \text{polylog}(1/\delta)$

In the above algorithm, $A(I)$ needs to be feasible with prob 1. Sometimes this is relaxed as well.

Working Definition

A randomized apprx algo is an α-factor apx for a (minimization) problem Π if

$$\mathbb{E}[\text{cost}(A(I))] \leq \alpha \cdot OPT(I)$$

Example 1 : (MAX-CUT)

Input :- Undirected Weighted Graph G

Output :- $S \subseteq V$

Obj :- Maximize $w(\delta S)$; $\delta S := \{ e \mid |e \cap S| = 1 \}$

Algorithm :

For every vertex v, put v in S w.p. $1/2$

Analysis

- S be the random output of the algorithm.
- $X_{uv} = 1$ if $(u,v) \in \delta S$, 0 o/w
- $ALG = \sum_{(u,v)} X_{u,v}$

- $P[X_{u,v} = 1] = Pr[u \in S, v \notin S] + Pr[u \notin S, v \in S]$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$$\therefore \mathbb{E}[ALG] = \frac{m}{2} \geq \frac{OPT}{2}$$

Linearity

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of expectation.

Brushing up some prob. facts

(1) Given any $X_1, \cdots, X_n$ rvs which are NOT NECESSARILY INDEPENDENT, we have

$$\mathbb{E}\left[X_1 + \cdots + X_n\right] = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots + \mathbb{E}X_n$$

(2) Two variables X, Y are independent if $\forall x, y$

$$\mathbb{P}\left[X = x \ \& \ Y = y\right] = \mathbb{P}\left[X = x\right] . \mathbb{P}\left[Y = y\right]$$

For independent random variables,

- $$\mathbb{E}\left[XY\right] = \mathbb{E}\left[X\right] \cdot \mathbb{E}\left[Y\right]$$
- $$\mathrm{Var}\left(X_1 + \cdots + X_n\right) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$$

(3) (Markov's Inequality)

For X which is ≥ 0, we have

$$\mathbb{P}\left[X \geq t\right] \leq \frac{\mathbb{E}X}{t}$$

(4) (Chebyshev's Inequality)

For any X,
$$\mathbb{P}\left[|X - \mathbb{E}X| \geq t \right] \leq \frac{\operatorname{Var} X}{t^2}$$

Set-Cover via Randomized Rounding

LP for SC:
$$\min \sum_{i=1}^{m} c(S_i)\, x_i \qquad =: LP$$
$$\forall j \text{ elt}: \sum_{i : j \in S_i} x_i \geq 1$$
$$x_i \in [0,1]$$

Algorithm
- Solve LP to get opt soln x
- For each set S_i, select it w.p. (independently)
$$p_i = \max(1, 3\ln n \cdot x_i)$$
- If some elt j is uncovered @ the end of the for loop in the prev. set, pick the cheapest containing j

Analysis:

- Given any j, let S_j^* be cheapest set cont. j

Observe: $c(S_j^*) \leq LP$

- The algo picks sets in two phases.

Let $X_i = 1$ if S_i is picked in stage 1
$= 0$ o/w

Let $Z_i = 1$ if S_i is picked in stage 2
$= 0$ o/w

- $ALG = \underbrace{\sum_{i=1}^{m} c(S_i)\, X_i}_{A_1} + \underbrace{\sum_{i=1}^{m} c(S_i)\, Z_i}_{A_2}$

- $P[X_i = 1] \leq 3 \ln n \cdot x_i$

$\therefore \mathbb{E}[A_1] \leq 3 \ln n \cdot LP$

- $Z_i = 1$ only if $S_i = S_j^*$ for some j

$\therefore$

$$Z_i = \begin{cases} 0 & \text{if } S_i \neq S_j^* \\ 1 & \text{if some } j \text{ st } S_i : S_j^* \text{ is left uncovered in stage 1} \end{cases}$$

$\therefore \Pr[Z_i = 1] \leq \Pr[\exists j : j \text{ is uncov. in stage 1}]$

(Union Bnd) $\leq \sum_{j=1}^{n} \mathbb{P}[j \text{ is uncov in stage 1}]$

$$\sum^{n} \prod_{i:\ldots} (1 - p_i)$$

$$= \sum_{j=1} \prod_{i:j\in S_i} (1-p_i)$$

$$\leq \sum_{j=1}^{n} \prod_{i:j\in S_i} (1 - 3x_i \cdot \ln n)$$

$$\leq \sum_{j=1}^{n} e^{-\sum_{i:j\in S_i} (3x_i \cdot \ln n)}$$

$$\leq \sum_{j=1}^{n} e^{-3\ln n} \qquad \because \sum_{i:j\in S_i} x_i \geq 1$$

$$\leq \frac{1}{n^2}$$

$$\therefore \mathbb{E}[A_2] \leq \sum_{i=1}^{n} c(S_i)\, \mathbb{E} z_i$$

$$\leq \frac{1}{n^2} \cdot n \cdot LP \leq \frac{LP}{n}$$

$$\therefore \mathbb{E}[ALG] \leq 3 \cdot \ln n \cdot LP$$

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Independent Set

Input: Undirected Graph $G = (V, E)$; $w : V \to \mathbb{R}_{\geq 0}$

O/p :- Independent set $I \subseteq V$

Obj :- Maximize $w(I)$

LP relxn :- $\max \sum_{v \in V} w_v x_v$

$$\forall (u,v) \in E : \quad x_u + x_v \leq 1$$

$$\forall u \in V : \quad x_u \in [0,1]$$

Algorithm

- Solve LP to get soln x
- If $W := \max_v w_v > \frac{LP}{2\sqrt{m}}$, return the max wt vertex.

else
- Independently sample each v prob. $p_v = \frac{x_v}{\sqrt{m}}$ and add to I'
- If $\exists (u,v) \in I'$ s.t. $(u,v) \in E$, delete BOTH from I'. Return $I \to$ what remains.

Analysis

- Let I_1 be the set obtained after step 2 and let $I_2 \subseteq I_1$ be the set removed

$$ALG = w(I_1) - w(I_2)$$

- $X_v = 1$ if $v \in I_1$; 0 o/w
- $Z_{uv} = 1$ if $u, v \in I_1$; 0 o/w

$$ALG = \sum_{v \in V} w_v X_v - \sum_{(u,v) \in E} (w_u + w_v) Z_{uv}$$

$$\therefore \mathbb{E}[ALG] = \sum_{v \in V} w_v \mathbb{E}[X_v] - \sum_{(u,v) \in E} (w_u + w_v) \mathbb{E}[Z_{uv}]$$

- $\mathbb{E}[X_v] = \Pr[X_v = 1]$

$$= \frac{x_v}{\sqrt{m}}$$

- $\mathbb{E}[Z_{uv}] = \mathbb{P}[X_v = 1 \ \& \ X_u = 1]$

$$= p_u p_v$$

$$= \frac{x_u x_v}{m}$$

$$\underset{\text{AM-GM}}{\leq} \frac{x_u^2 + x_v^2}{\,} \leq \frac{x_u + x_v}{\,}$$

$$\underset{\text{AM-GM}}{\leq} \frac{x_u^2 + x_v^2}{2m} \underset{\because x_u \leq 1}{\leq} \frac{x_u + x_v}{2m}$$

$$\underset{\text{LP-Constraint}}{\leq} \frac{1}{2m} \quad \text{for all } (u,v) \in E$$

$$\therefore \mathbb{E}[ALG] = \frac{1}{\sqrt{m}} \sum_v w_v x_v - \underbrace{\frac{1}{2m} \cdot \sum_{(u,v)\in E} (w_u + w_v)}_{\circledast}$$

$$\circledast = \frac{1}{2m} \cdot \sum_v w_v \cdot d_v$$

$\leftarrow$ wtd average of w_v's

$$\leq W_{max}$$

$$\leq \frac{LP}{2\sqrt{m}} \quad \ldots\ldots \text{ o/w If cond}^n \text{ of algo is true.}$$

$$\therefore \mathbb{E}[ALG] \geq \frac{LP}{\sqrt{m}} - \frac{LP}{2\sqrt{m}} = \frac{LP}{2\sqrt{m}}$$

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