CS 31: Algorithms (Spring 2018)

Not for Submission

A student intending to take this course should get 80% on the problem set below. This is only for your reference and not to be submitted, and the 80% is not set in stone.

Problem 1. ☕(**7 points**)

1. As the ancient parable goes, a king is approached by a priest with an 8×8 chess board who asks him to put a grain of rice on a corner square, two grains on the next, four on the next, eight on the next, and so on doubling the number of rice grains till all the 64 squares are filled. How many grains of rice will this king need?
   (a) Around a million
   (b) Around a trillion
   (c) More than number of atoms in the universe
   (d) None of the above
2. Given a *sorted* array of a billion numbers, you are asked whether 1729 is in the array or not. You can pay a dollar and ask for the value of the array in any position. How many dollars would you need to answer the question?
   (a) A billion
   (b) Around a million
   (c) Less than 100
   (d) Less than 20
3. A stream of integers is being sent to you by a system – for instance, the first 6 of these can be $(1023, 87, -98, 43, 56, 0)$. At any point, the system can ask whether a certain number, say 1729, is in the system or not. You want to be able to answer these questions very fast. Furthermore, whenever you see a new number you want to spend as little time as possible storing the number. What data structure would you use to store the stream of numbers?
4. Given a piece of code you want to figure out whether there any input would cause it to run in an infinite loop or not. How hard or easy do you think this problem is?
5. If I roll a die, what is the probability I see a prime number? (1 is not prime).
6. If I roll two dice, what is the probability the total is an odd number?
7. If I roll a die 5 times and I get 6 each time, what is the probability I will get a 6 if I roll it a sixth time?

Problem 2. ☕(**7 points**) For each of the following, state whether the statement is true or false. No proof needed.

1. $p \Rightarrow q \;\equiv\; \neg q \Rightarrow \neg p$

2. $\neg(p \land q) \;\equiv\; \neg p \lor \neg q$

3. $\neg(p \lor q) \;\equiv\; \neg p \land \neg q$

4. $p \land (q \lor r) \;\equiv\; (p \land q) \lor (p \land r)$ ($\land$ distributes over $\lor$)

5. $p \lor (q \land r) \;\equiv\; (p \lor q) \land (p \lor r)$ ($\lor$ distributes over $\land$)

6. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

8. $(A - B) - C = (A - C) - (B - C)$

9. $(A \cap B) \cup (A \cap \overline{B}) = A$

10. $(A - B) \cup (B - A) = A \cup B$

11. $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

12. $|A \cup B| = |A| + |B| - |A \cap B|$

13. If $f : A \to B$ is 1-to-1 for finite sets A and B, then $|A| \leq |B|$.

14. If $f : A \to B$ and $S \subseteq A$, then $f^{-1}(f(S)) = S$.

    (To interpret the above statement, here is a review of definitions. Let $f : A \to B$ be a function. We extend the definition of f as follows: for any $S \subseteq A$, $f(S) = \{f(x) \mid x \in S\}$. We call $f(S)$ the image of S under f. For any $T \subseteq B$, the inverse image of T, denoted $f^{-1}(T)$, is defined as $\{x \in A \mid f(x) \in T\}$.)

Problem 3. (2 + 1 + 4 points)

1. Given any simple undirected graph G, **prove** that the sum of degrees of the vertices is always even. ☕

2. Prove that in any graph the number of vertices of odd degree must be even. ☕

3. Recall that in a graph, a sequence $(u_1, u_2, \ldots, u_k)$ of vertices is a *path* if (u_i, u_{i+1}) is an edge for all $1 \leq i \leq k - 1$. u is reachable from v in the graph if there is a path starting at v and ending at u.

   Prove that in any graph and any odd-degree vertex v in the graph, there must exist another odd-degree vertex u in the graph such that v is reachable from u.

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Problem 4. ☕☕(3 + 4 points)

1. For any number $n \geq 1$ define the function

$$f(n) := 1 + 3 + \cdots + (2n-1) = \sum_{1 \leq i \leq n} (2i-1)$$

   Prove *using induction* that $f(n) = n^2$.

2. The Fibonacci numbers are defined as $F_0 = 1, F_1 = 1$ and $F_i = F_{i-1} + F_{i-2}$ for all $i > 1$. Prove *using induction* that $2^{n/2} \leq F_n \leq 2^n$ for all $n \geq 2$.

Problem 5. ☕☕☕(7 points)
Given any positive integer $n > 0$, define

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$$

Prove *using induction* that $H(n) \leq \log_2(n+1)$.
Hint: You may need the binomial theorem, which to remind you is the following : for any *real numbers* a, b and positive integer $n > 0$,

$$(a+b)^n = a^n + n \cdot a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1} + b^n$$