CS 30: Discrete Math in CS (Winter 2019): Lecture 11

Date: 23rd January, 2019 (Wednesday)

Topic: Propositional Logic

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.
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1. **Atomic Propositions/ Boolean Variables.** A *proposition* is a statement which takes one of the two *Boolean* values $\{\text{true}, \text{false}\}$. Here are a few examples.

   (a) $p : (2 + 2 = 4)$.

   (b) $q :$ (Nairobi is the capital of the USA).

   (c) $r :$ (The integer variable x takes value 1 in the run of this code.).

   Clearly, p is a proposition which takes value true, and q is a proposition which takes a value false. r is a proposition regarding a certain code, and its value will be determined by investigating the code. As of now, it is a *Boolean variable* whose value itself is a function of the integer variable x.

2. **Compound Propositions/ Boolean Formulas.** One can form compound propositions by taking atomic propositions and joining them together using operations. For instance, the following statement: "Either $2 + 2 = 4$, or Nairobi is the capital of the USA" contains a *either...or...* of two atomic statements. One of them is true, one of them is false, but since one is true it renders the compound statement, true.

   Compound Propositions are obtained by doing operations on Boolean Variables, and are often referred to as *Boolean Formulas*.

3. **Logic Operators.**

   - **Negation.** Given a proposition p, the proposition $q = \neg p$ is defined to take the value true if p takes the value false, and vice-versa, that is, q takes the value false if p takes the value true.

     For example, if $p : (2 + 2 = 4)$, then $q = \neg p$ is defined as $q : (2 + 2 \neq 4)$.

   - **OR and AND.** Given two atomic propositions p, q:
     - The proposition $p \lor q$ is true if and only if *at least one (and possibly both) of* p and q take the value true.
     - The proposition $p \land q$ takes the value true if and only if *both* p and q take the value true.

     For example, if p : (7 is even) and $q =$ (14 is even), then
     - $p \lor q$ is true since q is true.
     - $p \land q$ is false since p is false.

Exercise: *Suppose we had three atomic propositions p, q, r. When would $p \vee (q \vee r)$ be* true*? When would $p \wedge (q \wedge r)$ be true?*

- **Implications.** The final operation we look at is *implications.* The proposition $p \Rightarrow q$ is supposed to capture the proposition stating "if p is true, then q is true". The definition is that $p \Rightarrow q$ has truth value false *only if* p has truth value true and q has truth value false.

  For example, consider the statement "If it rains tomorrow, then I will not deliver mail tomorrow" Suppose p is the proposition "It will rain tomorrow" and q is the proposition "I will not deliver mail tomorrow.", then the above statement is captured by the proposition $(p \Rightarrow q)$.

  Both p and q are propositions (not quite math propositions). "Tomorrow" will decide what the values p and q take. If it rains, p takes the value true, otherwise it takes the value false. If I do deliver mail tomorrow, q takes the value false, otherwise it takes the value true.

  What about the proposition r though? Well, if it does rain tomorrow and I don't deliver mail, then r takes the value true. And, if it rains tomorrow but I do deliver mail, then r takes the value false. But the slightly interesting situation is if it *doesn't* rain tomorrow (that is, if p takes the value false). Suppose, furthermore, you did *not* deliver mail either (so q takes the value true.) What value do you think r takes? It takes the value true – the implication "still holds"; if the premise is false, then I can make *any* statement I want.

  Another example: The statement "If the sun rises in the West, then I am Batman" is *true*. It doesn't *matter* whether I am Batman or not; the sun doesn't rise in the West, and so I can say any garbage after "If the sun rises in the West,..." and the implication is still "correct". Useless, but correct. To contrast this, consider the slightly different statement "If the sun rises in the East, then I am Batman". Well this, if I read it, is false. Sun does rise in the East, and I am not Batman; ergo, the implication is untrue. Of course, if Batman reads it, he would think it is true.

  Formally, $(p \Rightarrow q)$ is true *unless* p takes the value true and q takes the value false.

4. **Truth Tables.** This is perhaps the most important construct in this lecture. Given a compound proposition, one can completely understand it by looking at the *truth table*, that is, the value this compound proposition takes given the possible settings of the underlying atomic propositions.

   Below are the truth tables of the various operations above.

p	$q = \neg p$
true	false
false	true

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Exercise: *Write the truth tables of:*

- $p \vee (q \vee r)$ *and* $p \wedge (q \wedge r)$.

p	q	$p \vee q$
true	true	true
true	false	true
false	true	true
false	false	false

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

- $\neg p \vee q$
- $p \wedge (q \vee r)$, *and* $(p \vee q) \wedge (p \wedge r)$.

5. **Order of Operations.** Just as in arithmetic, with logical operations there is an order in which they are applied. If the parantheses are not provided, then the first precedence is given to $\neg$. Then comes the ORs and ANDs – these are always parenthesized. And the last in order are implications.

   For example, the compound proposition $p \Rightarrow p \vee q$ actually means $p \Rightarrow (p \vee q)$ instead of $(p \Rightarrow p) \vee q$. Similarly, $\neg p \vee q$ means $(\neg p) \vee q$ and not $\neg(p \vee q)$. Generally, when in doubt put parenthesis.

6. **Logical Equivalence.** Two compound propositions/formulas are *logically equivalent* if they have the same truth tables. Here is an important example which expresses the $\Rightarrow$ using OR and negations.

   $$p \Rightarrow q \quad \equiv \quad \neg p \vee q \tag{Implication as OR}$$

   The proof of the above equivalence is described by the following truth table.

p	q	$p \Rightarrow q$	$\neg p$	$\neg p \vee q$
true	true	true	false	true
true	false	false	false	false
false	true	true	true	true
false	false	true	true	true

7. **Important Equivalences.** There are many important equivalences which one should internalize. They are listed below. You should (a) first get a feeling of these using plain English, and (b) then formally check **all** of them by writing truth tables. Think of this as one big exercise.

   - (**Negation of Negation.**) $\neg(\neg p) \equiv p$.

- (**Operation with** $\mathsf{true}, \mathsf{false}$**.**) $p \wedge \mathsf{true} \equiv p$; $p \vee \mathsf{true} \equiv \mathsf{true}$; $p \wedge \mathsf{false} \equiv \mathsf{false}$; $p \vee \mathsf{false} \equiv p$.
- (**Idempotence.**) $p \wedge p \equiv p$; $\quad p \vee p \equiv p$.
- (**Operation with Negation.**) $p \wedge \neg p \equiv \mathsf{false}$; $\quad p \vee \neg p \equiv \mathsf{true}$.
- (**Irrelevance.**) $p \vee (p \wedge q) \equiv p$; $\quad p \wedge (p \vee q) \equiv p$.
- (**Commutativity.**)
  - $p \vee q \equiv q \vee p$.
  - $p \wedge q \equiv q \vee p$.
- (**Associativity.**)
  - $p \vee (q \vee r) \equiv (p \vee q) \vee r$.
  - $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.
- (**Distributivity.**)
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
- (**Implications as an OR.**) $p \Rightarrow q \equiv \neg p \vee q$.
- (**De Morgan's Law.**) $\neg(p \vee q) \equiv \neg p \wedge \neg q$; $\quad \neg(p \wedge q) \equiv \neg p \vee \neg q$.

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Exercise: *Is* $(p \vee q) \wedge r \equiv p \vee (q \wedge r)$*?*