Probability: Linearity of Expectation[1]

- **Expectation of a Random Variable.** Let us recall the definition of the expectation of a random variable from last time.

$$\mathbf{Exp}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbf{Pr}[\omega] = \sum_{k \in \mathbb{R}} k \cdot \mathbf{Pr}[X = k]$$

- **Linearity of Expectation.** This is one of the most powerful equations in all of probability. Literally. It states the following. It literally has a four line proof.

Theorem 1. For any two random variables X and Y, let $Z := X + Y$. Then,

$$\mathbf{Exp}[Z] = \mathbf{Exp}[X] + \mathbf{Exp}[Y]$$

Proof.

$$\begin{aligned}
\mathbf{Exp}[Z] &= \sum_{\omega \in \Omega} Z(\omega)\,\mathbf{Pr}[\omega] && \text{Definition of Expectation} \\
&= \sum_{\omega \in \Omega} (X(\omega) + Y(\omega))\,\mathbf{Pr}[\omega] && \text{Definition of } Z \\
&= \sum_{\omega \in \Omega} X(\omega)\,\mathbf{Pr}[\omega] + \sum_{\omega \in \Omega} Y(\omega) \cdot \mathbf{Pr}[\omega] && \text{Distributivity} \\
&= \mathbf{Exp}[X] + \mathbf{Exp}[Y] && \text{Definition of Expectation}
\end{aligned}$$

□

As a corollary, by applying the above again and again $k-1$ times, we get:

Theorem 2. For any k random variables $X_1, X_2, \ldots, X_k$,

$$\mathbf{Exp}\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} \mathbf{Exp}[X_i]$$

Examples of applications.

a. *We roll two fair dice. X takes the value of the sum. In this case, $X = Y + Z$ where Y, Z are random variables of the kind from the previous bullet point.*
   Tailor-made application. $\mathbf{Exp}[Y] = \mathbf{Exp}[Z] = 3.5$, the expected value of a single roll of a die. Thus, $\mathbf{Exp}[X] = \mathbf{Exp}[Y + Z] = 7$ by linearity of expectation.

b. *We have a biased coin which lands heads with probability p. We toss it 100 times. Let Z be the number of heads we see. What is $\mathbf{Exp}[Z]$? Note that earlier we had the question for $p = 0.5$.*

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 28th Aug, 2021*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

> **Remark:** *Try doing this the "first-principle" way. That is, for each $0 \le k \le 100$, figure out the probability $\mathbf{Pr}[X = k]$ (that is, the probability we get exactly k heads), and then sum $\sum_{k=0}^{100} k \cdot \mathbf{Pr}[X = k]$. Please try it; feel the sweat needed to do this. It will make you appreciate the next three lines more!*

Define new random variables; define X_i to take the value 1 if the ith toss is heads, and 0 otherwise. Note, $X = X_1 + X_2 + \cdots + X_{100}$. Note, $\mathbf{Exp}[X_i] = p$ (it is a Bernoulli random variable). Thus, linearity of expectation gives $\mathbf{Exp}[X] = 100p$.

c. *n people checked in their hats, but on their way out, were handed back hats randomly. What is the expected number of people who get their correct hats?*

Define X_i to be 1 if the ith person gets his or her back correctly. What is $\mathbf{Exp}[X_i]$? It is $1/n$; it is the probability that $\sigma(i) = i$ for a random ordering σ. This question was there in the UGP. Let $Z = \sum_{i=1}^{n} X_i$. Note, Z is the number of people who get their correct hats. By linearity of expectation, $\mathbf{Exp}[Z] = 1$.

d. *In a party of n people there are some pairs of people who are friends, and some pairs who are not. In all there are m pairs of friends. The host randomly divides the party by taking each person and sending them left or right at the toss of a fair coin. How many friends, in expectation, are sundered apart?*

> **Remark:** *In terms of graphs (which we will see soon) the question is: a graph with m edges is randomly partitioned. How many edges, in expectation, have endpoints in different parts?*

For each pair of friends (u, v), define X_{uv} which takes the value 1 if u and v are split, and takes the value 0 if u and v are not split. The probability u and v are split is $1/2$ (either u is sent left, v is sent right, or vice-versa – do you see this?). Thus, $\mathbf{Exp}[X_{uv}] = 1/2$. Define $Z = \sum_{(u,v):\text{ friends}} X_{uv}$; Z is the number of friends sent apart. $\mathbf{Exp}[Z] = \sum_{(u,v):\text{ friends}} \mathbf{Exp}[X_{uv}] = m/2$. In expectation, half the friendships are sundered apart.

e. *In an ordering σ of $(1, 2, \ldots, n)$, an inversion is a pair $i < j$ such that $\sigma(i) > \sigma(j)$. How many inversions, in expectation, are there in a random permutation?*

Let σ be a random permutation. Define the *indicator random variable* X_{ij} for $i < j$, which takes the value 1 if $\sigma(i) > \sigma(j)$, and 0 otherwise. Note that $\mathbf{Pr}[X_{ij} = 1] = \frac{1}{2}$; there are equally many orderings with $\sigma(i) > \sigma(j)$ as $\sigma(i) < \sigma(j)$. Now note that $Z = \sum_{i=1}^{n} \sum_{j>i} X_{ij}$ is the number of inversions in σ. Thus, $\mathbf{Exp}[Z] = \sum_{i=1}^{n} \sum_{j>n} \mathbf{Exp}[X_{ij}] = \frac{1}{2} \cdot \frac{n(n-1)}{2}$.

- **Independent Random Variables.** Two random variables X and Y are independent, if for any $x \in \mathrm{range}(X)$ and any $y \in \mathrm{range}(Y)$,

$$\mathbf{Pr}[X = x, Y = y] = \mathbf{Pr}[X = x] \cdot \mathbf{Pr}[Y = y]$$

Examples:

– If we roll two dice, and X_1 and X_2 indicate the value of the rolls, then X_1 and X_2 are independent.

- If we have two independent events $\mathcal{A}$ and $\mathcal{B}$, then their indicator random variables $\mathbf{1}_\mathcal{A}$ and $\mathbf{1}_\mathcal{B}$ are independent.
- Consider a random variable X taking value $+1$ if a toss of a coins is head, and -1 if its tails. Such random variables are called *Rademacher random variables*. Suppose we toss the coin twice and X_1 and X_2 are the corresponding random variables. Then X_1 and X_2 are independent.

A set of k random variables $X_1, \ldots, X_k$ are *mutually independent* if for any $x_1, x_2, \ldots, x_k$ with $x_i \in \mathrm{range}(X_i)$, we have

$$\mathbf{Pr}[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k] = \prod_{i=1}^{k} \mathbf{Pr}[X_i = x_i]$$

Theorem 3. If X and Y are two independent random variables, then

$$\mathbf{Exp}[XY] = \mathbf{Exp}[X] \cdot \mathbf{Exp}[Y]$$

Proof.

$$\begin{aligned}
\mathbf{Exp}[XY] &= \sum_{x \in \mathrm{range}(X), y \in \mathrm{range}(Y)} (xy) \cdot \mathbf{Pr}[X = x, Y = y] && \text{Definition of Expectation} \\
&= \sum_{x \in \mathrm{range}(X), y \in \mathrm{range}(Y)} (xy) \cdot \mathbf{Pr}[X = x] \cdot \mathbf{Pr}[Y = y] && \text{Independence} \\
&= \left(\sum_{x \in \mathrm{range}(X)} x \cdot \mathbf{Pr}[X = x] \right) \cdot \left(\sum_{y \in \mathrm{range}(Y)} y \cdot \mathbf{Pr}[Y = y] \right) && \text{Algebra} \\
&= \mathbf{Exp}[X] \cdot \mathbf{Exp}[Y] && \text{Definition of Expectation}
\end{aligned}$$

□

Of course, there is no need to stick to two random variables. The theorem easily generalizes (do you see how?) to mutually independent random variables as follows.

Theorem 4. If $X_1, X_2, \ldots, X_k$ are mutually independent random variables, then

$$\mathbf{Exp}\left[\prod_{i=1}^{k} X_i\right] = \prod_{i=1}^{k} \mathbf{Exp}\left[X_i\right]$$

Examples.

- Let X_i and X_j be two independent Rademacher random variables. Recall, X_i takes $+1$ with probability $1/2$ and -1 with probability $1/2$. Then note (a) $\mathbf{Exp}[X_i] = \mathbf{Exp}[X_j] = 0$, (b) $\mathbf{Exp}[X_i \cdot X_i] = \mathbf{Exp}[X_j \cdot X_j] = 1$, and (c) $\mathbf{Exp}[X_i X_j] = \mathbf{Exp}[X_i] \cdot \mathbf{Exp}[X_j] = 0$. This is a very useful fact.

- Consider rolling a die n times, independently. Let Z be the random variable indicating the *product* of all the numbers seen. What is $\mathbf{Exp}[Z]$? To solve this, let X_i be the roll of the ith die. We know that $\mathbf{Exp}[X_i] = 3.5$ for all i. We also know $X_1, X_2, \ldots, X_n$ are all independent random variables. Thus, $\mathbf{Exp}[Z] = (3.5)^n$.

- **Some Famous Random Variables.**

  a. ***Bernoulli Random Variable.*** A random variable $X \sim \mathsf{Ber}(p)$ is a Bernoulli Random Variable with parameter p if $X = 1$ with probability p and 0 with probability $1 - p$. Basically, a coin toss where the probability of "heads" (with $X(\text{heads}) = 1$) is p instead of being $1/2$.

     Recall, given any event $\mathcal{E}$, the indicator random variable $\mathbf{1}_{\mathcal{E}}$ takes the value 1 if the event occurs and 0 otherwise (more precisely, $\mathbf{1}_{\mathcal{E}}(\omega) = 1$ for $\omega \in \mathcal{E}$ and 0 otherwise); indicator random variables are Bernoulli random variables with parameter $p = \mathbf{Pr}[\mathcal{E}]$.

     The expectation $\mathbf{Exp}[X]$ of $X \in \mathsf{Ber}(p)$ is precisely p.

  b. ***Binomial Random Variable.*** A random variable $X \sim \mathsf{Bin}(n, p)$ is the number of "heads" seen when one tosses n coins independently, where each coin comes heads with probability p. In other words, X is a sum of n independent Bernoulli random variables with probability p. The "shape" of this random variable is precisely given by

     $$\text{For integer } 0 \leq k \leq n, \quad \mathbf{Pr}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

     The expectation, however, is easy to calculate using linearity of expectation : $\mathbf{Exp}[X] = np$ since it's a sum of n Bernoulli's and each Bernoulli has expectation p.

  c. ***Geometric Random Variable.*** A random variable $X \sim \mathsf{Geom}(p)$ with parameter p, is the number of *times* a coin whose probability of heads is p needs to be tossed before we see the first heads. This is an interesting random variable whose range is *unbounded* (unlike the previous two); that is, given any integer k, there is a finite probability that $X > k$. Indeed, here is the "shape" (see you understand this)

     $$\text{For integer } k \geq 1, \quad \mathbf{Pr}[X = k] = \underbrace{(1-p)^{k-1}}_{\text{the first } (k-1) \text{ tosses are tails}} \cdot \underbrace{p}_{\text{and the } k\text{th toss is heads}}$$

     In the last lecture, we saw a random variable which counted the number of while-loops in a snippet of code. Check that it was an example of a geometric random variable. What was its parameter?

- **Expectation of a Geometric Random Variable.** One could calculate the expectation of $X \sim \mathsf{Geom}(p)$ "directly" via a calculation of the form

  $$\mathbf{Exp}[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$$

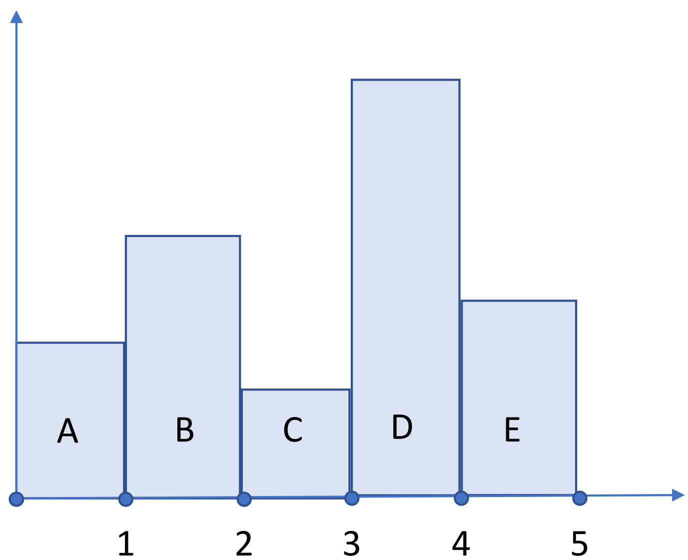
  This "smells" of the sum of a geometric series. To remind every one, for any number $0 < a < 1$, the sum $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$. If the above summation didn't have the extra "k" multiplying the thing inside the summation, we could apply the above. However, one can still figure out the summations like above analytically. Instead, I want to take this opportunity to show another interpretation of expectation which is also useful to keep in mind.

Theorem 5. For any positive integer valued random variable X, one has

$$\mathbf{Exp}[X] = \sum_{k=1}^{\infty} \mathbf{Pr}[X \geq k]$$

Remark: *One has the same for any random variable except if the range is not discrete the summation must be replaced by an integration.*

Proof. The proof is best illustrated by a picture. In the picture below, we are plotting the histogram/"shape" of the random variable. The x-axis has the positive integers. The y-axis has the probabilities. Now note that the area of the rectangle whose bottom right corner is at the integer k is *precisely* $\mathbf{Pr}[X = k]$. This is because the gap between k and $k-1$ is 1. In particular, the picture below shows only 5 rectangles presumably because the range of X is $\{1, 2, 3, 4, 5\}$ and the probability $\mathbf{Pr}[X = 1]$ is the area A, the probability $\mathbf{Pr}[X = 2]$ is the area B, and so on.



Note that the sum of the area $A + B + C + D + E$ is precisely 1. This is because the sum of the probabilities is 1. In particular, the area under this histogram is 1.

What is the expectation? Well, it is a weighted sum : $\mathbf{Exp} = A + 2B + 3C + 4D + 5E$. The key is to note that this sum can be thought of as taking all 5 rectangles, then taking all but the first, then taking all but the first two, and so on. More precisely,

$$\mathbf{Exp} = (A + B + C + D + E) \; + \; (B + C + D + E) \; + \; (C + D + E) \; + \; (D + E) \; + \; E$$

And now note the 5 different sums precisely correspond to $\mathbf{Pr}[X \geq k]$; for instance, $C + D + E = \mathbf{Pr}[X \geq 3]$. And this proves the theorem, at least pictorially. Now that we "feel" the proof, let's go and write it precisely.

$$\begin{aligned}
\mathbf{Exp}[X] &= \sum_{k=1}^{\infty} k \cdot \mathbf{Pr}[X = k] && \text{Definition of Expectation} \\
&= \sum_{k=1}^{\infty} k \cdot (\mathbf{Pr}[X \geq k] - \mathbf{Pr}[X \geq k+1]) && \text{An algebraic manipulation} \\
&= \sum_{k=1}^{\infty} \mathbf{Pr}[X \geq k] \cdot (k - (k-1)) && \text{Rearranging} \\
&= \sum_{k=1}^{\infty} \mathbf{Pr}[X \geq k] && (1)
\end{aligned}$$

The algebraic manipulation uses the fact that the random variable is integer valued. □

Now, we can use this to evaluate the expectation of a Geometric random variable.

Theorem 6. Let $X \sim \mathsf{Geom}(p)$ be a geometric random variable with parameter p. Then, $\mathbf{Exp}[X] = \frac{1}{p}$.

Proof. First let us observe that $\mathbf{Pr}[X \geq k] = (1-p)^{k-1}$. Why is this? Well the event $\{X \geq k\}$ is **equivalent** to the first $(k-1)$ tosses coming in tails. Make sure you see this. Therefore,

$$\mathbf{Exp}[X] = \sum_{k \geq 1} (1-p)^{k-1} = \frac{1}{p}$$

where we directly used the sum of geometric series. □