Functions[1]

- **Definition.** A function is a *mapping* from one set to another. The first set is called the *domain* of the function, and the second set is called the *co-domain*. For *every* element in the domain, a function assigns a *unique* element in the co-domain.

  Notationally, this is represented as

  $$f : A \to B$$

  where A is the set indicating the domain $\mathrm{dom}(f)$, and B is the set indicating the co-domain $\mathrm{codom}(f)$. For every $a \in A$, the function maps the value of $a \mapsto f(a)$ where $f(a) \in B$.

  The *range* of the function is the subset of the co-domain which are *actually mapped to*. That is, $b \in B$ is in the range if and only if there is some element $a \in A$ such that $f(a) = b$. The range can also be written in the set-builder notation as

  $$\mathrm{range}(f) := \{f(a) : a \in A\}$$

  Remark: *For any function f with finite domains and ranges, we have* $|\mathrm{range}(f)| \le |\mathbf{dom}(f)|$

- **An Example.** Suppose

  $A = \{1, 2, 3\}$, and $B = \{5, 6\}$, then the map $f(1) = 5, f(2) = 5, f(3) = 6$ is a valid function.

  A is the domain. B is the co-domain. In this example, B also happens to be the range.

- **The Identity Function.** When the domain is the same as the co-domain, the *idenitity* function $\mathrm{id} : A \to A$ maps $a \in A$ to $a \mapsto a$.

- **More Examples.**

  - Usually (say in calculus) a function is described as a formula like $f(x) = x^2$. Henceforth, whenever you see a function ask your self how does it map to the above definition. In this example, this is as follows.

    the domain is $\mathbb{R}$, the set of real numbers, and so is the co-domain. The map is $x \mapsto x^2$ – check both are real numbers. The range of the function is the set of non-negative real numbers (sometimes denoted as $\mathbb{R}_+$).
  - $f(x) = \sin x$ is a function whose domain is $\mathbb{R}$ and the range is the interval $[-1, 1]$.
  - A (deterministic) computer program/algorithm is also a function; its domain is the set of possible inputs and its range is the set of possible outputs.

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 28th Aug, 2021*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

Remark: *How about the function $f(x) = \sqrt{x}$? Is this a function? When you think about it, you see some issues if we don't define the domain and co-domain. For instance, if the domain contains* negative *numbers, then what is $\sqrt{-1}$? Ok, so perhaps the domain is all positive real numbers. However, we also have a problem with $\sqrt{4}$ – is it mapping to $+2$ or -2? Note it can only map to a* unique *number. This can be resolved by stating the domain and co-domain are both non-negative reals, and the $x \mapsto \sqrt{x}$ goes to the positive root.*

Exercise: *Given a set $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, describe a function $f : A \to B$ whose range is $\{5\}$, and describe a function g whose range is $\{4, 6\}$. Just to get a feel, how many functions can you describe of the first form (whose range is $\{5\}$), and how many functions can you describe of the second form?*

- **Sur-, In-, Bi- jective functions.** A function $f : A \to B$ is
  - *surjective*, if the range is the same as the co-domain. That is, for every element $b \in B$ there exists some $a \in A$ such that $f(a) = b$. Such functions are also called *onto* functions.

    For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, and consider the function $f : A \to B$ with $f(1) = 5$, $f(2) = 5$, and $f(3) = 6$. Then, f is surjective. This is because for $5 \in B$ there is $1 \in A$ such that $f(1) = 5$ and for $6 \in B$ there is a $3 \in A$ such that $f(3) = 6$.

    Remark: *If A and B are* finite sets*, and $f : A \to B$ is a surjective function, then $|B| \leq |A|$?*

  - *injective*, if there are no collisions. That is, for any two elements $a \neq a' \in A$, we have $f(a) \neq f(a')$. Such functions are also called *one-to-one* functions.

    For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$, and consider the function $f : A \to B$ with $f(1) = 5$, $f(2) = 6$, and $f(3) = 8$. Then, f is injective. This is because $f(1), f(2), f(3)$ are all distinct numbers.

    Remark: *If A and B are* finite sets*, and $f : A \to B$ is an injective function, then $|A| = |\text{range}(f)|$. Thus, $|A| \leq |B|$.*

    Injective functions have *inverses*. Formally, given any injective function $f : A \to B$, we can define a function $f^{-1} : \text{range}(f) \to A$ as follows

    $$f^{-1}(b) = a \quad \text{where } a \text{ is the } \textit{unique } a \in A \text{ with } f(a) = b.$$

  - *bijective*, if the function is both surjective and injective.

    For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the function $f(x) = 2x$ defined over the domain A and co-domain B is a bijective function. Can you see why?

    Remark: *If A and B are* finite sets*, and $f : A \to B$ is a bijective function, then $|B| = |A|$. We will see this useful fact many times in the combinatorics module.*

- **Composition of Functions.** Given a function $f : A \to B$ and a function $g : B \to C$, one can define the composition of g and f, denoted as $g \circ f$ with domain A and co-domain C as follows:

$$(g \circ f)(a) = g(f(a)) \quad \text{that is} \quad a \mapsto g(f(a))$$

  Note this is well defined since for every $a \in A$, $f(a) \in B$, and thus $g(f(a)) \in C$.

  Examples

  - Suppose $A = \{1, 2, 3\}$ and $B = \{5, 6\}$ and $C = \{3, 4\}$. Also suppose $f : A \to B$ is defined as $f(1) = 5$, $f(2) = 6$, and $f(3) = 5$; and $g : B \to C$ is defined as $g(5) = 3$ and $g(6) = 4$, then the composed function is $(g \circ f)(1) = 3$, $(g \circ f)(2) = 4$, and $(g \circ f)(3) = 3$.
  - If $f : \mathbb{R} \to \mathbb{R}_+$ defined as $f(x) = x^2$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $g(x) = \sqrt{x}$ (as defined above), then (convince yourself) that $(g \circ f)(x)$ returns the *absolute* value of x.
  - If $f : A \to B$ is a bijection, and $f^{-1} : B \to A$ is its inverse, convince yourself that $(f^{-1} \circ f) : A \to A$ is the $\mathrm{id} : A \to A$ identity function.

Answers to exercises

- $f(1) = 5, f(2) = 5, f(3) = 5$ is an example of $f : A \to B$ with range $\{5\}$. Similarly, $g(1) = 4, g(2) = 4, g(3) = 6$ is an example of a function $g : A \to B$ with range $\{4, 6\}$. There is only one function of the first type. Of the second type there are more. For instanct $h(1) = 4, h(2) = 6, h(3) = 6$ also has range $\{4, 6\}$. Can you find one more? How many such functions are there?