Propositional Logic[1]

- **Atomic Propositions/ Boolean Variables.** A *proposition* is a statement which takes one of the two *Boolean* values $\{\mathsf{true}, \mathsf{false}\}$. Here are a few examples.

  a. $p : (2 + 2 = 4)$.

  b. q : (Nairobi is the capital of the USA).

  c. r : (It will rain sometime tomorrow.).

  Clearly, p is a proposition which takes value true, and q is a proposition which takes a value false. r is a proposition regarding the future, and its value will be determined tomorrow. As of now, it is a *Boolean variable*.

- **Compound Propositions/ Boolean Formulas.** One can form compound propositions by taking atomic propositions and joining them together using operations. For instance, the following statement: "Either $2 + 2 = 4$, or Nairobi is the capital of the USA" contains a *either...or...* of two atomic statements. One of them is true, one of them is false, but since one is true it renders the compound statement, true.

  Compound Propositions are obtained by doing operations on Boolean Variables, and are often referred to as *Boolean Formulas*.

- **Logic Operators.**

  – **Negation.** Given a proposition p, the proposition $q = \neg p$ is defined to take the value true if p takes the value false, and vice-versa, that is, q takes the value false if p takes the value true.

    For example, if $p : (2 + 2 = 4)$, then $q = \neg p$ is defined as $q : (2 + 2 \neq 4)$.

  – **OR and AND.** Given two atomic propositions p, q:

    * The proposition $p \vee q$ is true if and only if *at least one (and possibly both) of* p and q take the value true.
    * The proposition $p \wedge q$ takes the value true if and only if *both* p and q take the value true.

    For example, if p : (7 is even) and $q =$ (14 is even), then

    * $p \vee q$ is true since q is true.
    * $p \wedge q$ is false since p is false.

  – **Implications.** The final operation we look at is *implications.* The proposition $p \Rightarrow q$ is supposed to capture the proposition stating "if p is true, then q is true". The definition is that $p \Rightarrow q$ has truth value false *only if* p has truth value true and q has truth value false.

    For example, consider the statement "If it rains tomorrow, then I will not deliver mail tomorrow" Suppose p is the proposition "It will rain tomorrow" and q is the proposition "I will not deliver mail tomorrow.", then the above statement is captured by the proposition $(p \Rightarrow q)$.

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 28th Aug, 2021*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

Both p and q are atomic propositions. "Tomorrow" will decide what the values p and q take. If it rains, p takes the value true, otherwise it takes the value false. If I do deliver mail tomorrow, q takes the value false, otherwise it takes the value true.

What about the proposition $r := (p \Rightarrow q)$ though? Well, if it does rain tomorrow and I don't deliver mail, then r takes the value true. And, if it rains tomorrow but I do deliver mail, then r takes the value false. But the slightly interesting situation is if it *doesn't* rain tomorrow (that is, if p takes the value false). Suppose, furthermore, you did *not* deliver mail either (so q takes the value true.) What value do you think r takes? It takes the value true – the implication "still holds"; if the premise is false, then I can make *any* statement I want.

Another example: The statement "If the sun rises in the West, then I am Batman" is *true*. It doesn't *matter* whether I am Batman or not; the sun doesn't rise in the West, and so I can say any garbage after "If the sun rises in the West,..." and the implication is still true. Useless, but true. To contrast this, consider the slightly different statement "If the sun rises in the East, then I am Batman". Well this, if I read it, is false. Sun does rise in the East, and I am not Batman; ergo, the implication is untrue. (Of course, if Batman reads it, he would think it is true.)

Formally, $(p \Rightarrow q)$ is true *unless* p takes the value true and q takes the value false.

- **Truth Tables.** Given a compound proposition, one can completely understand it by looking at the *truth table*, that is, the value this compound proposition takes given the possible settings of the underlying atomic propositions.

  Below are the truth tables of the various operations above.

p	$\neg p$
true	false
false	true

p	q	$p \vee q$
true	true	true
true	false	true
false	true	true
false	false	false

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

Exercise: *Write the truth tables of:*

– $p \vee (q \vee r)$ *and* $(p \vee q) \vee r$.

– $p \vee (q \wedge r)$, *and* $(p \vee q) \wedge (p \vee r)$.

- **Order of Operations.** Just as in arithmetic, with logical operations there is an order in which they are applied. If the parantheses are not provided, then the first precedence is given to $\neg$. Then comes the ORs and ANDs – these are always parenthesized. And the last in order are implications.

  For example, the compound proposition $p \Rightarrow p \vee q$ actually means $p \Rightarrow (p \vee q)$ instead of $(p \Rightarrow p) \vee q$. Similarly, $\neg p \vee q$ means $(\neg p) \vee q$ and not $\neg(p \vee q)$. Generally, when in doubt put parentheses.

- **Logical Equivalence.** Two compound propositions/formulas are *logically equivalent* if they have the same truth tables. Here is an important example which expresses the $\Rightarrow$ using OR and negations.

$$p \Rightarrow q \quad \equiv \quad \neg p \vee q \qquad \text{(Implication as OR)}$$

  The proof of the above equivalence is described by the following truth table.

p	q	$p \Rightarrow q$	$\neg p$	$\neg p \vee q$
true	true	true	false	true
true	false	false	false	false
false	true	true	true	true
false	false	true	true	true

- **Important Equivalences.** There are many important equivalences which one should internalize. They are listed below. You should (a) first get a feeling of these using plain English, and (b) then formally check **all** of them by writing truth tables. Think of this as one big exercise.

  – (**Negation of Negation.**) $\neg(\neg p) \equiv p$.
  – (**Operation with** true, false**.**) $p \wedge \mathsf{true} \equiv p$; $p \vee \mathsf{true} \equiv \mathsf{true}$; $p \wedge \mathsf{false} \equiv \mathsf{false}$; $p \vee \mathsf{false} \equiv p$.
  – (**Idempotence.**) $p \wedge p \equiv p$; $\quad p \vee p \equiv p$.
  – (**Operation with Negation.**) $p \wedge \neg p \equiv \mathsf{false}$; $\quad p \vee \neg p \equiv \mathsf{true}$.
  – (**Irrelevance.**) $p \vee (p \wedge q) \equiv p$; $\quad p \wedge (p \vee q) \equiv p$.
  – (**Commutativity.**)
    * $p \vee q \equiv q \vee p$.
    * $p \wedge q \equiv q \wedge p$.
  – (**Associativity.**)
    * $p \vee (q \vee r) \equiv (p \vee q) \vee r$.
    * $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.
  – (**Distributivity.**)
    * $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
    * $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
  – (**Implications as an OR.**) $p \Rightarrow q \equiv \neg p \vee q$.
  – (**De Morgan's Law.**) $\neg(p \vee q) \equiv \neg p \wedge \neg q$; $\quad \neg(p \wedge q) \equiv \neg p \vee \neg q$.

- **Tautologies, Contradictions, and Satisfiability.**

  A formula ϕ is a *tautology* if it takes the truth value true *no matter* what values the underlying variables take. That is, ϕ is logically equivalent to true. We have already see one tautology: $p \vee \neg p$ in the operation with negation.

  Here is another example

$$\phi := \quad p \wedge (p \Rightarrow q) \Rightarrow q$$

  One way to check this the truth table.

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$p \wedge (p \Rightarrow q) \Rightarrow q$
true	true	true	true	true
true	false	false	false	true
false	true	true	false	true
false	false	true	false	true

  Another way is to reduce the formula as follows to prove that the formula is equivalent to true.

$$\begin{aligned}
p \wedge (p \Rightarrow q) \Rightarrow q &\equiv p \wedge (\neg p \vee q) \Rightarrow q && \text{Implication as OR} \\
&\equiv ((p \wedge \neg p) \vee (p \wedge q)) \Rightarrow q && \text{Distributivity} \\
&\equiv (\mathsf{false} \vee (p \wedge q)) \Rightarrow q && \text{Operation with Negation} \\
&\equiv (p \wedge q) \Rightarrow q && \text{Operation with } \mathsf{false} \\
&\equiv \mathsf{true} && \text{Exercise}
\end{aligned}$$

  A formula ϕ is a *contradiction* or *unsatisfiable* if it takes the truth value false *no matter* what values the underlying variables take. That is, ϕ is logically equivalent to false.

  Exercise: *Prove that the following formula is a contradiction*

$$\Big((\neg p \wedge q) \vee (p \wedge \neg q)\Big) \wedge (p \Rightarrow q) \wedge (q \Rightarrow p)$$

  A formula ϕ is *satisfiable* if there is some setting of the underlying variables which makes it true. That is, it is *not* unsatisfiable.

Answers to exercises.

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p	q	r	$(q \vee r)$	$p \vee (q \vee r)$	$(p \vee q)$	$(p \vee q) \vee r$
true	true	true	true	true	true	true
true	true	false	true	true	true	true
true	false	true	true	true	true	true
true	false	false	false	true	true	true
false	true	true	true	true	true	true
false	true	false	true	true	true	true
false	false	true	true	true	false	true
false	false	false	false	false	false	false

p	q	r	$(q \vee r)$	$p \wedge (q \vee r)$	$(p \wedge q)$	$(p \wedge r)$	$(p \wedge q) \vee (p \wedge r)$
true	true	true	true	true	true	true	true
true	true	false	true	true	true	false	true
true	false	true	true	true	false	true	true
true	false	false	false	false	false	false	false
false	true	true	true	false	false	false	false
false	true	false	true	false	false	false	false
false	false	true	true	false	false	false	false
false	false	false	false	false	false	false	false

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$$\begin{aligned}
\Big((\neg p \wedge q) \vee (p \wedge \neg q)\Big) \wedge (p \Rightarrow q) \wedge (q \Rightarrow p) \quad & \underbrace{\equiv}_{\text{Implication as OR}} \Big((\neg p \wedge q) \vee (p \wedge \neg q)\Big) \wedge (\neg p \vee q) \wedge (\neg q \vee p) \\
& \underbrace{\equiv}_{\text{De Morgan's Law}} \Big(\neg(p \vee \neg q) \vee \neg(\neg p \vee q)\Big) \wedge (\neg p \vee q) \wedge (\neg q \vee p) \\
& \underbrace{\equiv}_{\text{De Morgan's Law}} \neg\Big((p \vee \neg q) \wedge (\neg p \vee q)\Big) \wedge \Big((\neg p \vee q) \wedge (\neg q \vee p)\Big) \\
& \underbrace{\equiv}_{\text{Operation with Negation}} \mathsf{false}
\end{aligned}$$