Graphs : The Ford-Fulkerson Algorithm[1]

In the last lecture we showed that the maximum s, t flow is *at most* the minimum s, t cut. Furthermore we looked at the conditions which would prove max-flow equals min-cut. Let's recall that theorem for we will use this later.

Theorem 1. Suppose f is a feasible s, t flow f, and S is an s, t cut S such that

a. $f(e) = u(e)$ for all $e \in \partial^+ S$
b. $f(e) = 0$ for all $e \in \partial^- S$

Then f is a ***maximum*** s, t flow, S is a ***minimum*** s, t cut, and their values are the same.

In this lecture, we will prove the ***strong duality*** theorem: in any network, the maximum value of an s, t flow equals the capacity of the minimum s, t cut. We do so via an *algorithm*. That is, we describe an algorithm which in one swoop solves both the MAX-s, t-FLOW and the MIN-s, t-CUT problem, and also proves their respective values are the same. This algorithm was designed by Lester Ford and Dilbert Ray Fulkerson in the 1950s, and is called the Ford-Fulkerson algorithm. To describe this, we first introduce the concept of the *residual networks*.

1 The Residual Network

Let us start with an algorithm for finding maximum flows that *doesn't* work. Recall what we need to do: we need to find a valid flow $f : E \to \mathbb{R}_{\geq 0}$ such that $\mathsf{excess}_f(t)$ is maximized. We start with the zero flow: $f(e) = 0$ for all $e \in E$, and try to *increase* this flow in iterations. Now consider an s, t-path p in the graph G. Given such a path p, we can *augment* the current flow f *along the path* p as follows:

- Let $\delta = \min_{e \in p} u(e)$
- For every $e \in p$, set $f(e) \leftarrow f(e) + \delta$.

Note that flow conservation remains valid; the total in-flow at any $v \neq s, t$ is equal to the total out-flow – it is either δ or 0. Also note that by choice of δ and since we started from the 0-flow, the capacity constraint also remains valid. Finally, the $\mathsf{excess}_f(t)$ increases by δ. Progress!

How should we proceed? We could repeat the steps above, namely, find another s, t-path p and then augment flow along path p. However, we have already sent some flow which could have used up some capacity of certain edges e. In the augmentation step we should be wary of this lest we violate the capacity constraint. The fix is to maintain a ***residual capacity*** $u_f(e)$ for every edge e. These are initially set to $u(e)$, the original capacity, but for every unit of flow that we pass through this edge, we must *decrease* its residual capacity. This leads to the following augmentation procedure along path p *given* we have sent flow f:

- Let $\delta = \min_{e \in p} u_f(e)$
- For every $e \in p$, set $f(e) \leftarrow f(e) + \delta$.

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 19th Mar, 2022*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

- For every $e \in p$, set $u_f(e) \leftarrow u_f(e) - \delta$.

The above process can be repeated over and over again, and every time the value of the flow increases by δ. We stop when $\delta = 0$, that is, we can't find any path p from s to t with $\min_{e \in p} u_f(e) > 0$. How would we check this? Simple: remove all edges with $u_f(e) = 0$ and check if there is a path from s to t. We write the full algorithm below.

procedure NAIVEMAXFLOW(G, s, t, u):
  Start with $f \equiv 0$ and $u_f(e) = u(e)$ for all e.
  ▷ *Invariant:* $u_f(e) + f(e) = u(e)$ *for all* e.
  while true **do**:
    Find any path p from s to t with $\min_{e \in p} u_f(e) =: \delta > 0$.
    If no such path **break**
    For every edge $e \in p$: $f(e) \leftarrow f(e) + \delta$; $u_f(e) \leftarrow u_f(e) - \delta$.
  return f

As can be guessed by the name and the color of the shading, the algorithm above, although a solid try, doesn't return the correct solution. Let's see an example where it fails (maybe you'd like to try to find one first before peeking?): see Figure 1.

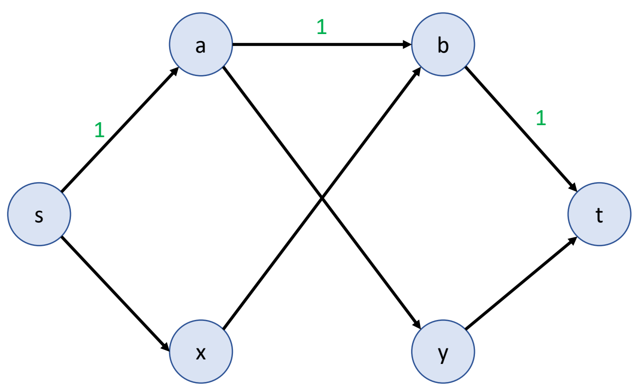

Figure 1: In this graph G, all edges have unit capacity. If we send our first augmentation along the path $p = (s, a, b, t)$, then we would send 1 unit of flow on this. All these edges would have $u_f(e) = 0$ and deleting these edges disconnects s and t. Thus the NAIVEMF algorithm would terminate. On the other hand, there is a flow of value 2 which sets $f(e) = 1$ for all edges except (a, b). This would have been achieved if we sent flow first on the path (s, x, b, t) and then (s, a, y, t). But how would we know to do that?

In a sense, the flow we chose to send, that is the one on the path (s, a, b, t) was a *mistake*. The main idea behind the notion of the residual network is to keep safeguards which help us correct mistakes when made. This is a general life principle, but something which beautifully works in the case of s, t flows.

Definition 1. Given a flow network (G, s, t, u) and a valid flow $f : E \to \mathbb{R}_{\geq 0}$, the *residual network* with respect to flow f denoted as G_f is defined as follows:

- $G_f = (V, E_f)$ where $E_f = E \cup E_{\text{rev}}$

- $E_{\text{rev}} = \{(v,u) : f(u,v) > 0\}$, that is, E_{rev} contains the *reverse* of all edges which carry positive flow.
- The residual capacity on edges in E_f is defined as follows

$$u_f(x,y) = \begin{cases} u(x,y) - f(x,y) & \text{if } (x,y) \in E \\ f(y,x) & \text{if } (x,y) \in E_{\text{rev}} \end{cases}$$

Let us draw the reverse graph for the network in Figure 1 with respect to the flow of unit 1 sent along the path s, a, b, t. This is shown in Figure 2.

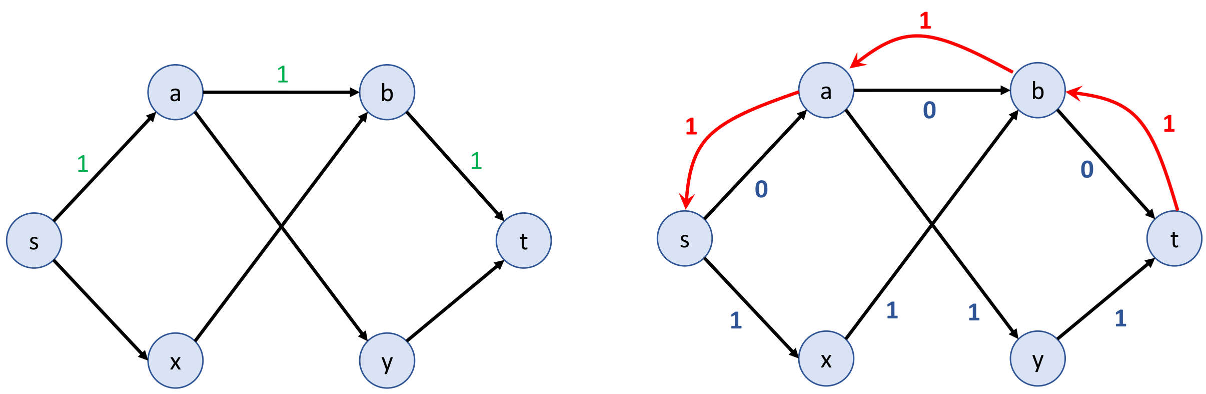

Figure 2: The graph in the left shows the flow in green. The graph in the right is the residual graph. The red edges are E_{rev}. The numbers are the residual capacities.

Why is the residual network important? Well, note that after the flow f is sent on the path (s,a,b,t), the residual network G_f *does* have a path from s to t where every edge has a residual capacity $u_f(e) > 0$; this path is $q = (s,x,b,a,y,t)$. As you can see, this path contains one edge (b,a) which is not in E but in E_{rev}.

The question that should come into your mind now is: so what? The edge (b,a) doesn't even exist in the graph G; why are we bothering with such abstract constructs? Well, suppose you suppressed those thoughts and tried to *augment* flow along this path q. (Wait! Firstly there is no edge (b,a) and now you are asking me to send flow across it?) But here's the point: we know that since $(b,a) \in E_{\text{rev}}$ there must exist $(a,b) \in E$ with $f(a,b) > 0$. Indeed, $f(a,b) = u_f(b,a)$. So *increasing* flow along the dummy reverse edge $(b,a) \in E_{\text{rev}}$ is actually just a short-hand for *decreasing* the flow along the edge (a,b). This augmentation is indicating that our first choice of sending flow across the edge (a,b) was perhaps a "mistake", and this is fixing it. Indeed, this is the conceptual abstraction of the residual network: send flow along edges, but keep the reverse edges as stop guards to rectify potential mistakes. Now we are ready to formally give the algorithm.

2 The Ford Fulkerson Algorithm

First, we formally define what augmentation along a path in a residual network means.

procedure AUGMENT(G_f, s, t, p):
  ▷ *Augment along path p in the residual network G_f.*
  ▷ *Modifies $f(e)$ for every $e \in G$; modifies $u_f(e)$ for every edge $e \in E_f$.*
  $\delta := \min_{e \in p} u_f(e)$.
  For every edge $e = (x, y) \in p$:

- If $(x, y) \in E$:
  - $f(x, y) \leftarrow f(x, y) + \delta$;
  - $u_f(x, y) \leftarrow u_f(x, y) - \delta$;
  - $u_f(y, x) \leftarrow u_f(y, x) + \delta$;
- If $(x, y) \in E_{\mathsf{rev}}$:
  - $f(y, x) \leftarrow f(y, x) - \delta$; ▷ *Note: $(y, x) \in E$*
  - $u_f(y, x) \leftarrow u_f(y, x) + \delta$;
  - $u_f(x, y) \leftarrow u_f(x, y) - \delta$;

The following invariants should be checked from the pseudocode above.

Claim 1 (Invariants of Augmentation)**.**

I1. For every edge $e \in E \cup E_{\mathsf{rev}}$, $u_f(e) \geq 0$
I2. For every edge $(x, y) \in E$, $f(x, y) + u_f(x, y) = u(x, y)$
I3. For every $(x, y) \in E_{\mathsf{rev}}$, $f(y, x) = u_f(x, y)$.

Claim 2. If f satisfied the capacity constraints before AUGMENT, then it does so after AUGMENT too.

Proof. This follows from the Invariants: For any edge $(x, y) \in E$, we have $f(x, y) = u(x, y) - u_f(x, y) \leq u(x, y)$ (from I2 and I1, respectively). Similarly, I1 implies $u_f(y, x) \geq 0$, that is, $f(x, y) \geq 0$. □

Claim 3. If f is an s, t flow in G which satisfies flow conservation constraints at every vertex $v \neq s, t$, then the flow after AUGMENT step also satisfies flow conservation constraints at every vertex $v \neq s, t$. Furthermore, $\mathsf{excess}_f(t)$ goes up by δ.

Proof. If $v \notin p$, then there is nothing to discuss. So assume $v \in p$. Since $v \notin \{s, t\}$ it is an internal node in p and let (w, v) and (v, x) be the two edges of p incident on it. There are four cases to consider.

- Case 1: $(w, v) \in E, (v, x) \in E$. In this case, both $f(w, v)$ and $f(v, x)$ go up by δ, implying the increase in excess is 0.
- Case 2: $(w, v) \in E, (v, x) \in E_{\mathsf{rev}}$. In this case, $f(w, v)$ goes up by δ and $f(x, v)$ goes down by δ, implying the increase in excess is 0.
- Case 3: $(w, v) \in E_{\mathsf{rev}}, (v, x) \in E$. In this case, $f(v, w)$ goes down by δ and $f(v, x)$ goes up by δ, implying the increase in excess is 0.
- Case 4: $(w, v) \in E_{\mathsf{rev}}, (v, x) \in E_{\mathsf{rev}}$. In this case, both $f(v, w)$ and $f(x, v)$ go down by δ, implying the increase in excess is 0.

Let $(v, t) \in p$ be the edge incident on t. If $(v, t) \in E$, then $f(v, t)$ increases by δ and the flow on no other edge incident on t changes, implying $\mathsf{excess}_f(t)$ goes by δ. If $(v, t) \in E_{\mathsf{rev}}$, then $f(t, v)$ decreases by δ and the flow on no other edge incident on t changes, implying $\mathsf{excess}_f(t)$ goes by δ. □

Now we are ready to describe the maximum flow algorithm.

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procedure FordFulkerson(G, s, t, u):
    Initialize f ≡ 0 and u_f ≡ u and G_f ≡ G.
    while true do:
        Check if there is an s,t path p in G_f with all u_f(e) = 0 edges removed.
        If not, break.
        Else, Augment(G_f, s, t, p).
    return (f, G_f).
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Lemma 1. If $u(e)$s are integer valued, then FORDFULKERSON returns an integer valued valid f in $O(nmU)$ time, where $U := \max_{e \in E} u(e)$.

Proof. Since the 0-flow is valid, and the Augmentation Claims imply AUGMENT maintains validity, we get that the final flow is valid. We claim that the Line 4 in AUGMENT will set δ to a positive integer valued. To see this, we need to prove u_f is integer valued. But this is true in the beginning (when $u_f \equiv u$), and since subsequently f is augmented in δ-installments, the f is always integral which in turn leads to u_f being integral. Furthermore, each time $\mathsf{excess}_f(t)$ grows by $\delta \geq 1$. Since the final flow is valid, the total value of this flow $\mathsf{excess}_f(t) \leq nU$ since there can be at most n edges of the form (v, t) and each has capacity at most U. Thus, the algorithm terminates in $O(nU)$ rounds. Finally, note each round takes $O(n + m)$ time. □

The next lemma proves the flow returned is a max-flow by showing that the conditions of Theorem 1 holds.

Lemma 2. The flow f returned by FORDFULKERSON when it terminates is a maximum valued flow.

Proof. We describe a cut induced by a subset S which satisfied the properties of the corollary. In fact, define

$$S = \{v : v \text{ is reachable from } s \text{ in } G_f \text{ with all } u_f(e) = 0 \text{ edges removed.}\}$$

Clearly, $s \in S$. Since the algorithm terminates, $t \notin S$.

Now fix an $(x, y) \in \partial^+(S)$. Since y is not reachable from s using positive residual capacity edges, we get $u_f(x, y) = 0$. By I2, this implies

$$\text{For } (x, y) \in \partial^+(S), \quad f(x, y) = u(x, y)$$

Now consider an $(x, y) \in \partial^-(S)$. Since x is not reachable from s using positive residual capacity edges, we get $u_f(y, x) = 0$ for $(y, x) \in E_{\mathsf{rev}}$. That is,

$$\text{For } (x, y) \in \partial^-(S), \quad f(x, y) = 0$$

But these are precisely the conditions of Theorem 1. Thus, f is a maximum s, t flow and S is a minimum s, t cut. In one swoop, FORDFULKERSON (+ one DFS) finds not only the max-flow but also the min-cut. □

Theorem 2. Given a flow network (G, s, t, u) where $u(e)$ is a positive integer for every edge $e \in E(G)$, the FORDFULKERSON algorithm finds a maximum s, t flow which is integral, and a minimum s, t cut in $O(nmU)$ time, and max s, t flow equals min s, t cut.